Basic Exam: Fall 2024

Test instructions:

- Write your UCLA ID number on the upper right corner of each page.
- Do not write your name anywhere on the exam.
- Your final score will be the sum of

  FIVE linear algebra problems (Problems 1-6) and

  FIVE analysis problems (Problems 7-12).

  However, to pass the exam you need to show mastery of both subjects.
- Indicate below which 10 problems you wish to have graded.
- Please staple your problems in numerical order.

1	2	3	4	5	6
7	8	9	10	11	12

Linear Algebra Problems

Problem 1. Let $A = \{a_{ij}\}_{i,j=1}^{d}$ be an integer-valued matrix whose row sums are all equal to a positive integer n. Noting that $\det(A)$ is an integer, prove that n divides $\det(A)$.

Problem 2. Let $n \geq 2$ be an integer and A be a normal complex-valued $n \times n$-matrix. Assume that $\dim(\mathrm{Ker}(\lambda - A)) \leq 1$ for all $\lambda \in \mathbb{C}$. Prove that any normal complex-valued $n \times n$-matrix B that commutes with A takes the form

$$B = b_0 \mathrm{Id} + b_1 A + \cdots + b_{n-1} A^{n-1}$$

for some $b_0, \ldots, b_{n-1} \in \mathbb{C}$.

Problem 3. Suppose that T is a linear operator on a finite-dimensional complex vector space with spectrum $\sigma(T) = \{0\}$. Show that T is nilpotent.

Problem 4. Consider the polynomial

$$p(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$$

(To save you some time, note that this polynomial vanishes only at $x = 1$ and $x = -2$.) Let A be a complex-valued 4×4-matrix such that $p(A) = 0$. Assume also that $\mathrm{Tr}(A) = 1$ and $\mathrm{Rank}(A - \mathrm{Id}) = 2$. Provide the Jordan canonical form for A. Justify your answer.

Problem 5. Solve the system of ODEs for functions $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= 2x(t) + y(t) \\ y'(t) &= x(t) - 2y(t) \end{aligned}$$

with initial data $x(0) = 1$ and $y(0) = 0$.

Problem 6. Let V be a linear vector space of functions on a set X of finite cardinality $|X|$. Prove that $\dim(V) \leq |X|$. Then prove existence of points $x_1, \ldots, x_n \in X$, with $n = \dim(V)$, and functions $f_1, \ldots, f_n \in V$ such that

$$f_i(x_j) = \delta_{ij}, \quad i, j = 1, \ldots, n$$

(In particular, $f_1, \ldots, f_n$ is a basis in V.)

Analysis Problems

Problem 7. Prove that given any continuous, increasing function $f\colon \mathbb{R} \to \mathbb{R}$ with

$$\lim_{x\to\infty} f(x) = \infty,$$

there exists a differentiable, strictly increasing function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to\infty} g(x) = \infty, \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

Problem 8. Let $a < b$ be reals. Prove that if $f\colon [a,b] \to \mathbb{R}$ is a continuous and one-to-one function, then f is monotone.

Problem 9. Consider two real-valued sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ which satisfy the following two properties:

(i) There exists a constant $A > 0$ such that for all $n \in \mathbb{N}$ the following holds:

$$\left|\sum_{i=1}^{n} a_i\right| \leq A$$

(ii) The sequence $\{b_n\}_{n\in\mathbb{N}}$ is monotone decreasing and $\lim_{n\to\infty} b_n = 0$.

Do the following:

(a) Show that, for all $m \in \mathbb{N}$,

$$\sum_{i=1}^{m} a_i b_i = F_m\, b_m - F_{n-1}\, b_n + \sum_{i=n}^{m-1} F_i(b_i - b_{i-1})$$

holds with $F_k := \sum_{i=1}^{k} a_i$.

(b) Use part (a) to prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Problem 10. Let X and Y be metric spaces, with distance functions d_X and d_Y, respectively. Prove that if X is compact, then any continuous function $f\colon X \to Y$ is uniformly continuous. For full credit, solve this problem directly from the relevant definitions, without reference to any major theorems.

Problem 11. (a) Suppose $F(x, y)$ is a continuous function on $\mathbb{R}^2$ such that for every rectangular domain $D = [a, b] \times [c, d]$ the following holds:

$$\iint_D F(x, y)dxdy = 0.$$

Prove that $F(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

(b) Assume that $f(x, y)$, $\frac{\partial}{\partial x} f(x, y)$, $\frac{\partial}{\partial y} f(x, y)$, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right)$ are all continuous on $\mathbb{R}^2$. Use part (a) of this problem to prove that:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right).$$

You may assume without proof that the iterated integrals $\int\int F(x, y)dxdy$ and $\int\int F(x, y)dydx$ are equal.

Problem 12. Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ such that

(i) f_n has at most one discontinuity point for each $n \in \mathbb{N}$, and

(ii) $f_n \to f$ uniformly for some function $f : \mathbb{R} \to \mathbb{R}$.

Prove that f has at most one discontinuity point.