QUALIFYING EXAM

Geometry/Topology

Fall 2024

Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $f\colon M \to N$ be a nonsingular smooth map between connected manifolds of the same dimension. Answer the following questions with a proof or counter-example.

   (a) Is f necessarily injective or surjective?

   (b) Is f necessarily a covering map when N is compact?

   (c) Is f necessarily an open map?

   (d) Is f necessarily a closed map?

2. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions m and n, respectively, such that $m+n=p$. Suppose that $M \cap N = \varnothing$. Consider the linking map

$$\lambda\colon M \times N \to S^p, \quad \lambda(x,y) = \frac{x-y}{\|x-y\|}.$$

   The degree of λ is called the linking number $l(M,N)$.

   (a) Show that $l(M,N) = (-1)^{(m+1)(n+1)} l(N,M)$.

   (b) Show that if M is the boundary of a compact oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M,N)=0$.

3. Let ω be a 1-form on a connected manifold M. Show that ω is exact, i.e., $\omega = df$ for some function f, if and only if for all piecewise smooth closed curves $c\colon S^1 \to M$ it follows that $\int_c \omega = 0$.

4. Let ω be a smooth, nowhere vanishing 1-form on a three-dimensional smooth manifold M^3.

   (a) Show that $\ker \omega$ is an integrable distribution on M if and only if $\omega \wedge d\omega = 0$.

   (b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. In standard coordinates, define the symmetric $(0,2)$-tensor Hessian to be

$$\mathrm{Hess} f = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j.$$

   Let $g(X,Y) = X \cdot Y$ denote the usual Euclidean inner product between tangent vectors $X, Y \in T_p\mathbb{R}^n$, and let $\mathscr{L}_{\nabla f} g$ denote the Lie derivative of the $(0,2)$-tensor g in the direction of the gradient vector field ∇f. Prove

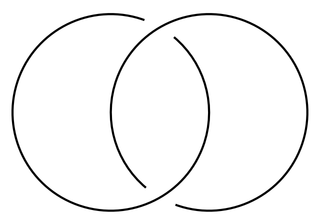
$$\mathrm{Hess} f = \frac{1}{2} \mathscr{L}_{\nabla f} g.$$

6. Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

7. Let $M = T^2 - D^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces arising as 3-fold covers of M, along with the covering maps.

8. Let $n > 0$ be an integer and let A be an abelian group with a finite presentation by generators and relations. Show that there exists a topological space X with $H_n(X) \cong A$.

9. Let $H \subset S^3$ be the Hopf link, shown in the figure



Compute the fundamental group and the homology groups of the complement $S^3 - H$.

10. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions, with relations $i^2 = j^2 = -1$, $ij = -ji = k$. The multiplicative group $\mathbb{H}^* = \mathbb{H} - \{0\}$ acts on $\mathbb{H}^n - \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^*$ is called the quaternionic projective space. Calculate its homology groups.