Logic

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple it to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1____________________	**5**____________________
2____________________	**6**____________________
3____________________	**7**____________________
4____________________	**8**____________________

Pass/Fail recommendation on this form.

Total score: ____________________

Qualifying Examination
LOGIC
Fall 2024

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1. Work with a theory T in a countable language $\mathcal{L}$, and let S_n be the topological space of complete n-types of T. Recall that the basic open sets of S_n are of the form $[\psi] = \{p : p \in S_n \text{ and } \psi \in p\}$, where ψ ranges over $\mathcal{L}$-formulas with n free variables.

(1a) Prove that if $[\psi]$ is uncountable, then there is a formula ϑ such that both $[\psi \wedge \vartheta]$ and $[\psi \wedge \neg\vartheta]$ are uncountable.

(1b) Prove that if $[\psi]$ is uncountable, then $|[\psi]| \geqslant 2^{\aleph_0}$.

Problem 2. Prove that being a wellordering is not a first-order property. That is, prove that there is no formula ϑ such that $(A; R) \models \vartheta$ if and only if R is a wellordering on A.

Problem 3. Let $(\varphi_e^X : e \in \omega)$ be the standard enumeration of partial recursive functions relative to an oracle $X \subseteq \omega$. Consider the sets

$$A := \left\{e \in \omega : \varphi_e^{\emptyset'}(e)\downarrow= 0\right\} \quad \text{and} \quad B := \left\{e \in \omega : \varphi_e^{\emptyset'}(e)\downarrow= 1\right\}.$$

Show that there is no Δ_2^0 set $C \subseteq \omega$ with $A \subseteq C \subseteq \omega \setminus B$.

Problem 4. Does there exist a complete consistent extension PA^* of PA such that $\mathsf{PA}^* \leqslant_T \emptyset'$?

Problem 5. Show that there exist $2^{\aleph_0}$ non-isomorphic countable models of PA.

Problem 6. Let $x \in 2^\omega$ be 1-*generic*, meaning that for each r.e. set W of finite binary strings, there is $n \in \omega$ such that either $x{\upharpoonright}n \in W$ or no string in W extends $x{\upharpoonright}n$. Let

$$A := \{k \in \omega : x(2k) = 1\} \quad \text{and} \quad B := \{k \in \omega : x(2k+1) = 1\}.$$

Show that these sets are Turing incomparable.

Problem 7. Suppose there is an inaccessible cardinal.

(7a) Prove that there is a countable ordinal α such that $(L_\alpha; \in) \models \mathsf{ZFC}$.

(7b) Let α be the least ordinal witnessing (7a). Let $(H; \in) \preceq (L_\alpha; \in)$. Prove that $H = L_\alpha$.

Problem 8. Work in ZF (without the axiom of choice). Show that the Axiom of Choice holds if and only if for every set X and every proper class $\mathcal{C}$, there exists an injection $f : X \to \mathcal{C}$.