ONLA

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers. Read the directions of the exam very carefully.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

****

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________
9. ____________________

Pass/fail recommend on this form.

Total score: ____________________

Qualifying Exam, Fall 2024

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

---

1. (10 points) Let A be a unitary matrix.
   a) Prove that the condition number of A is equal to 1.
   b) Prove that A is orthogonally diagonalizable.

2. (10 points) Let A be a real square matrix that has eigendecomposition $A = V\Lambda V^{-1}$ (here Λ is the diagonal matrix of eigenvalues and V is the nonsingular eigenvector matrix). Suppose that a perturbation $A + \delta A$ has eigenvalue μ. Prove that there exists some eigenvalue λ of A such that $|\lambda - \mu| \leq \kappa(V)\|\delta A\|$, where $\kappa(V)$ denotes the condition number of V and $\|\cdot\|$ the spectral norm.

   Hint: You may wish to first prove that if μ is not an eigenvalue of A, then -1 is an eigenvalue of $(\Lambda - \mu I)^{-1}V^{-1}\delta AV$.

3. (10 points) Assume the fundamental axiom of floating point arithmetic is in place.

    a) Prove or disprove that backwards substitution is backward stable for a 2×2 upper triangular system.

    b) Prove or disprove that the addition of 1 is backward stable (i.e. the algorithm defined by $\tilde{f}(x) = \mathrm{fl}(x) \oplus 1$ where $\oplus$ denotes floating point addition and $\mathrm{fl}(x)$ denotes the floating point representation of x).

4. (10 points) Consider $Ax = b$ with $A \in \mathbb{R}^{2\times 2}$ solved with Gauss-Seidel and Jacobi iterations.

   a) Derive the spectral radius for both methods.

   b) Prove that Gauss-Seidel converges if and only if Jacobi converges.

5. (10 points) Consider $Ax = b$:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

and the Conjugate Gradient (CG) algorithm

$$\begin{aligned} & r_0 = b - Ax_0, p_0 = r_0, \\ & \text{for } i = 0, 1, 2, \ldots \\ & \quad \alpha_i = (r_i^T r_i)/(p_i^T A p_i) \\ & \quad x_{i+1} = x_i + \alpha_i p_i \\ & \quad r_{i+1} = r_i - \alpha_i A p_i \\ & \quad \beta_i = (r_{i+1}^T r_{i+1})/(r_i^T r_i) \\ & \quad p_{i+1} = r_{i+1} + \beta_i p_i \end{aligned}$$

a) With kth Krylov subspace $\mathcal{K}_k = \text{span}\{r_0, Ar_0, \ldots, A^{k-1} r_0\}$, determine the vectors defining the Krylov spaces for $k \leq 3$, taking initial approximation $x_0 = 0$.

b) Solve $Ax = b$ with CG using zero initial guess $x_0 = 0$.

c) Verify that $r_0, \ldots, r_{k-1}$ form an orthogonal basis for $\mathcal{K}_k$ for $k = 1, 2, 3$.

d) Verify that $p_0, \ldots, p_{k-1}$ form an A-orthogonal basis for $\mathcal{K}_k$ for $k = 1, 2, 3$.

6. (10 points) Recall that the Lanczos iteration tridiagonalizes a hermitian A by building towards

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

The Lanczos algorithm is given by

$$\begin{aligned} &\beta_0 = 0, \quad q_0 = 0, \quad b = \text{arbitrary}, \quad q_1 = b/\|b\| \\ &\text{for } n = 1, 2, 3, \ldots \\ &\qquad v = Aq_n \\ &\qquad a_n = q_n^T v \\ &\qquad v = v - \beta_{n-1} q_{n-1} - \alpha_n q_n \\ &\qquad \beta_n = \|v\| \\ &\qquad q_{n+1} = v/\beta_n \end{aligned}$$

Assuming exact arithmetic, prove that during Lanczos iterations q_{j+1} is orthogonal to $q_1, q_2, \ldots, q_j$.

7. (10 points) Consider the problem

$$\text{extremize} \quad x_1 x_2 + x_1^2 \quad \text{subject to} \quad x_1^2 - 2 \leq x_2 \leq -x_1^2 + 2.$$

(a) Write down the KKT conditions for this problem and find all points that satisfy them.

(b) Determine whether or not the points in part (a) satisfy the second order necessary conditions for being local maximizers or minimizers.

(c) Determine whether or not the points that satisfy the necessary conditions in part (b) also satisfy the second order sufficient conditions for being local maximizers or minimizers.

Hint: Draw a rough sketch of the objective function and the constraints. Maximization of a function f can be treated as minimization of $-f$.

8. (10 points) Let $X \subseteq \mathbb{R}^n$ be a convex set and let $f, g_1, \ldots, g_m$ be convex functions over X. Assume there exists $\hat{x} \in X$ such that

$$g_1(\hat{x}) < 0, \ldots, g_m(\hat{x}) < 0.$$

Let $c \in \mathbb{R}$. Show that the following are equivalent (nonlinear Farkas lemma).

(a) The following implication holds:

$$x \in X, g_1(x) \leq 0, \ldots, g_m(x) \leq 0 \quad \Rightarrow \quad f(x) \geq c.$$

(b) There exist $\lambda_1, \ldots, \lambda_m \geq 0$ such that

$$\min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} \geq c.$$

Hint: One direction is easy. For the other direction you may use the following result on the separation of two convex sets: Let $C_1, C_2 \subseteq \mathbb{R}^n$ be two nonempty convex sets with $C_1 \cap C_2 = \emptyset$. Then there exists $a \in \mathbb{R}^n, a \neq 0$ with $a^T x \leq a^T y$ for any $x \in C_1$, $y \in C_2$.

9. (10 points) Let $f \in C_L^{1,1}(\mathbb{R}^n)$ and assume that $\nabla^2 f(x) \geq 0$ (positive semi-definite) for all $x \in \mathbb{R}^n$. Suppose that the optimal value of the problem $\min_{x\in\mathbb{R}^n} f(x)$ is f^*. Let $\{x_k\}_{k\geq 0}$ be the sequence generated by the gradient descent method with constant stepsize $\frac{1}{L}$. Show that if $\{x_k\}_{k\geq 0}$ is bounded, then $f(x_k) \to f^*$ as $k \to \infty$.

   Hint: $C_L^{1,1}(\mathbb{R}^n)$ denotes the set of continuously differentiable functions on $\mathbb{R}^n$ whose gradient satisfies $\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2$ for all $x, y \in \mathbb{R}^n$.