Algebra Qualifying Exam

Read the instructions of the exam carefully.

Complete this sheet and staple to your answers.

STUDENT ID NUMBER ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW

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1. ________________________
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Total score: ____________________

Pass/fail is recommended on this form.

ALGEBRA QUALIFYING EXAM

SEPTEMBER 17, 2024

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

Complete 8 of the 10 problems. Clearly indicate, by circling the numbers of the problems below, which 8 problems should be graded.

$$\boxed{1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10}$$

Throughout the exam, justify your answers.

No books, notes, phones, or other printed or electronic materials can be used on the exam.

Please staple your problems in the order they are listed in the exam.

Problem 1. Let G be a finite group and k a field of characteristic p dividing the order of G. Is there any such example with an isomorphism of kG-modules $kG \cong M_1 \oplus M_2$ such that M_1 has dimension one over k?

Problem 2. Let p, q be distinct prime numbers and consider the number field $K = \mathbb{Q}(\sqrt{p} + \sqrt{q})$. Describe all subfields of K and inclusions between them.

Problem 3. Let R be a commutative ring and $S \subset R$ a multiplicatively closed subset. Construct a natural transformation (in either direction) between the functors $\operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$ and $S^{-1}\operatorname{Hom}_R(M, N)$, considered as functors of R-modules M and N. Prove that your natural transformation is an isomorphism if M is finitely presented.

Problem 4. Let K be a field and $f\colon \mathrm{M}_m(K) \to \mathrm{M}_n(K)$ a K-linear ring homomorphism, on rings of matrices of size $m, n \geq 1$. Prove that $m \leq n$.

Problem 5. Let $A = \mathbb{R}[X, Y]/(Y^2 - X^2(X + 1))$.

(1) Prove that A is a domain.
(2) Suppose that $A \subseteq B$ is an integral extension, with $B \cong \mathbb{R}[Z_1, \ldots, Z_d]$ a polynomial ring over $\mathbb{R}$. What is d?

Problem 6. Let G be the group of 3×3 complex matrices of the form

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with nonzero entries on the diagonal. (The group operation is matrix multiplication.) Show that G is solvable.

Problem 7. Let $F = \mathbb{Q}(\sqrt[3]{5})$. Show that for every field E containing $\mathbb{Q}$, the ring $F \otimes_{\mathbb{Q}} E$ is either a field, a product of two fields, or a product of three fields. Give examples of all three cases, justifying your answer.

Problem 8. Let R be a commutative ring and let M be an R-module. Suppose that the functor $F(X) := \operatorname{Hom}_R(M, X)$ from R-modules to R-modules has a right adjoint. Show that the R-module M is finitely generated. [You may use general results from category theory, but you need to show that M is finitely generated in the usual sense for an R-module.]

Problem 9. Let F be a field of characteristic $\neq 2$, and let $a, b \in F^*$. Let A be the associative F-algebra generated by elements i, j with the relations $i^2 = a$, $j^2 = b$, and $ij = -ji$. You may use that A has dimension 4 as an F-vector space. Show that A is a simple algebra with center F.

Problem 10. Let G be the (dihedral) group presented by

$$\langle a, b \mid a^5 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

You may use that G has order 10. Compute the character table of G.