Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1. ________________________ 5. ____________________________

2. ________________________ 6. ____________________________

3. ________________________ 7. ____________________________

4. ________________________ 8. ____________________________

Pass/fail recommend on this form.

Total score: ____________________

ADE Exam, Spring 2025
Department of Mathematics, UCLA

1. [10 points] For $x(t) \in \mathbb{R}^n$ and real $n \times n$ matrices A (a constant matrix) and $B(t)$, consider the system

$$\dot{x} = Ax + B(t)x + f(x)\,,$$

where

$$\sup_{t \in \mathbb{R}} \|B(t)\| \leq \delta$$

and f is locally Lipschitz, such that

$$\|f(x)\| \leq R\|x\|^2$$

holds for $\|x\| \leq 1$ and for some $R > 0$.

   (a) (5 pts) Assume that A is symmetric and negative definite. Show that the origin is an asymptotically stable equilibrium provided δ is chosen sufficiently small by studying the time derivative of $\|x(t)\|^2$ and showing for initial data sufficiently small that $\|x(t)\|^2$ is a Lyapunov function that exponentially decreases to 0 along trajectories.

   (b) (5 pts) This time, use a fixed-point argument to prove that you only need to assume that $\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ (where $\sigma(A)$ denotes the spectrum of A, i.e., the set of eigenvalues) to conclude that the origin is an asymptotically stable fixed point provided δ is sufficiently small.

2. [10 points] Derive power-series representations of the two linearly independent solutions of the following differential equation on $x > 0$:

$$y'' + \left(\frac{c}{x} + d\right) y = 0\,,$$

where c and d are real-valued constants.

3. [10 points] Consider the deconvolution problem $K * u + n = f$, where $K \in C_0^\infty(R^2)$ is a blurring kernel, n (a noise term) is continuous with compact support, $f \in C_0(R^2)$ is a blurred signal (e.g., blurred data), and we recall that a convolution has the form

$$A * B = \int A(x-y)B(y)\,dy\,.$$

To solve this inverse problem, we consider solving the following variational problem:

$$\min_{u \in H_0^1(R^2)} \left[\int_{R^2} |K * u - f|^2 \, dx \, dy + \int_{R^2} |\nabla u|^2 \, dx \, dy \right], \tag{1}$$

which minimizes the L^2 norm of the difference between the data f and the blurred signal $K * u$ plus the H^1 seminorm of the signal.

(a) (5 pts) Compute the first variation of the energy in (1).

[As a reminder, the first variation arises from looking at perturbations ϵv of the signal u and linearizing in powers of ϵ.]

Write the answer as a nonlocal elliptic problem of the form $\Delta u = \mathrm{NL}(u)$, where the nonlocal operator NL involves the convolution operator K.

[Here it may be helpful to recall the identity $\int gK * h = \int hK * g$.]

(b) (5 pts) Compute the Fourier transform of the variational problem (1) and write the solution in Fourier space in terms of Fourier modes.

4. [10 points]

(a) (5 pts) Characterize the region in $\mathbb{R}^2$ for which

$$Lu = -xu_{xx} + (2 + y)u_{xy} - 2u_{yy} \tag{2}$$

is uniformly elliptic.

(b) (5 pts) Find the smallest $c \in R$ for which L given by

$$Lu = -xu_{xx} + u_{xy} + u_{yy} \tag{3}$$

is uniformly elliptic in $\{(x, y) \in R^2 | x > c + \epsilon\}$ for all $\epsilon > 0$.

[For the 2nd-order PDE

$$-\sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j u) = f$$

with $x \in \Omega$, the associated condition of **uniform ellipticity** *is that*

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for some $\theta > 0$, all $x \in \Omega$, and all $\xi \in \mathbb{R}^n$.]

5. [10 points] Consider the semilinear heat equation

$$u_t = \Delta u + (1-u)u\,, \ \ x \in S^1 \text{ (the circle)}\,, \ t > 0\,. \tag{4}$$

   (a) (5 pts) Prove for any C^2 initial condition $u_0 \in (0,1)$ that if there exists a solution of (4) in the space $C^1[0,T] \cap C^2(S^1)$, then it satisfies $0 < u(x,t) < 1$ on the entire time interval $[0,T]$.

   (b) (5 pts) Prove that any solution of (4) that satisfies the conditions of part (a) is unique.

6. [10 points] Consider the heat equation

$$u_t - \Delta u = 0\,, \quad x \in U \subseteq \mathbb{R}^n\,, \ t > 0\,, \tag{5}$$

where U is an open set.

By considering dilation scaling

$$u(x,t) \mapsto \lambda^{\alpha} u(\lambda^{\beta} x, \lambda t) \tag{6}$$

for all $\lambda > 0$, $x \in \mathbb{R}^n$, $t > 0$, and appropriate α and β (which you will determine) and normalizing $\int_{\mathbb{R}} u\,dx = 1$, derive the so-called "fundamental solution" of (5).

7. [10 points] Consider the problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx}\,, \\ v(x,0) &= \phi(x)\,, \ v_t(x,0) = \psi(x)\,, \\ v(0,t) &= v(\ell,t) = 0\,. \end{aligned} \tag{7}$$

Derive the solution $v(x,t)$, which you should express as a sum of traveling waves, and draw an associated space-time diagram that clearly conveys domains of dependence and wave reflections in this system.

8. [10 points] Solve

$$x^2 \psi_x + xy\psi_y = \psi^2 \tag{8}$$

for $\psi(x,y)$, subject to the boundary condition $\psi = 1$ on the curve Γ defined by $x = y^2 \neq 0$.

Sketch the characteristics that pass through Γ.

Describe the set of points in the (x,y) plane that can be reached from the curve Γ by following characteristics.