Analysis Qualifying Exam, March 27, 2025

Warning: Please, read carefully the new rules as they have slightly changed since the last Analysis Qualifying Exam.

Rules: Complete four problems in Real Analysis and four problems in Complex Analysis, and <u>circle</u> their numbers below – otherwise, the first 4 problems will be graded. You must demonstrate adequate knowledge of both real analysis (Problems 1–6) and complex analysis (Problems 7–12). A **complete solution** of a problem is preferable to partial progress on several problems.

1 2 3 4 5 6 7 8 9 10 11 12

Problem 1. Given $f \in L^1(\mathbb{R})$, compute $\lim_{t \to +\infty} \int_{\mathbb{R}} |f(x-t) - f(x)| dx$. Justify your answer.

Problem 2. The two parts of this problem are not related.

a. Let $p \in (1, +\infty)$. Suppose that $(f_n)_n \subset L^p(0,1)$ and $\|f_n\|_{L^p} \leq 1$ for all n. Assuming that $f_n(x) \to 0$ a.e., prove that $(f_n)_n$ converges weakly to 0 in $L^p(0,1)$.

b. Suppose $f \in L^1(\mathbb{R})$ and $\lambda > 0$. Show that $\lim_{n \to +\infty} n^{-\lambda} f(nx) = 0$ for almost all $x \in \mathbb{R}$ (where $n \in \mathbb{N}$).

Hint. You can use that $\int f(nx)dx = \frac{1}{n} \int f(x)dx$ without further justification.

Problem 3. Suppose that $p \in (1, +\infty)$ and q is the dual exponent of p in the sense that $p^{-1} + q^{-1} = 1$. Let $K \in L^p((0,1)^2)$.

a. For $f \in L^q(0,1)$, show that $(Af)(x) := \int_0^1 K(x,y)f(y)dy$ exists for almost every x and show that A is a bounded linear operator of $L^q(0,1)$ to $L^p(0,1)$.

b. Suppose that for every $f \in L^q(0,1)$, $(Af)(x) = 0$ for almost every x. Show that $K = 0$ a.e.

Problem 4. Let p be a real number such that $1 \leq p \leq +\infty$.

a. Say whether or not the supremum of a family of continuous functions on $L^p(\mathbb{R})$ is either lower semicontinuous, upper semicontinuous or neither. Justify your response.

b. Let $A \subset L^p(\mathbb{R})$ denote those functions for which

$$\lim_{L \to +\infty} \int_0^L f(x)\, dx = 0.$$

Show that A is a Borel subset of $L^p(\mathbb{R})$.

Problem 5. Let $f \in L^1(0,1)$ be such that $\int_0^1 f(x)g^{(3)}(x)dx = 0$ for all $g \in C_c^\infty(0,1)$, where $g^{(3)}$ is the third derivative of g. Show that f is (almost everywhere) a polynomial of degree at most 2. (In fact, one could solve the problem for general n as there is nothing special about $n = 3$).

Hint: First step: approximate f by smooth functions using convolutions.

Problem 6. Show that for any $f \in L^2(\mathbb{R})$,

$$\lim_{n \to \infty} n \iint f(x)f(y)\, e^{-2n|x-y|}\, dx\, dy = \int_{\mathbb{R}} f(x)^2\, dx.$$

Problem 7. Fix $\theta > 0$ and consider the meromorphic function

$$F(z) = \frac{1}{\cosh(z) + \cosh(\theta)}.$$

a. Determine the location of all poles of $F(z)$ and their residues.
b. Determine all $p \in \mathbb{C}$ for which $F(z) = F(z+p)$.
c. Rigorously evaluate the following integral for all $\xi \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\cosh(x) + \cosh(\theta)}\, dx.$$

Problem 8. Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$. Suppose $f : \mathbb{C}_+ \to \mathbb{R}$ is harmonic, non-negative, and satisfies

$$\int_0^\infty \big|f(iy)\big|\, dy < \infty.$$

Show that f is identically zero.

Problem 9. Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ and $\overline{\mathbb{C}_+} = \{z : \operatorname{Im} z \geq 0\}$. Suppose $f : \overline{\mathbb{C}_+} \to \overline{\mathbb{C}_+}$ is continuous and satisfies

$$f(x) \in \mathbb{R} \quad \text{whenever} \quad x \in \mathbb{R}.$$

Show that if f is holomorphic in $\mathbb{C}_+$ and not constant there, then it maps $\mathbb{C}_+$ *onto* $\mathbb{C}_+$.

Problem 10. Consider the entire function

$$F(z) = 1 + \sum_{n=1}^{\infty} \left(\tfrac{z}{n}\right)^n$$

Show that for any $R > 0$, the function $F(z)$ has no more than $4R$ zeros (counting multiplicity) in the ball $\{z \in \mathbb{C} : |z| < R\}$.

Problem 11. Let $\Omega = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Construct an example of a holomorphic function $f : \Omega \to \mathbb{C}$ with the *both* of the following properties:
(i) *None* of the powers $f(z)^n$ of $f(z)$ with $n \in \mathbb{N}$ admits a primitive in Ω.
(ii) The reciprocal, $1/f(z)$ is holomorphic in Ω and *does* admit a primitive.

You must demonstrate that your example has both properties. Recall that $G(z)$ is a primitive of $g(z)$ if $G(z)$ is differentiable and $G'(z) = g(z)$.

Problem 12. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and define a function $A : (0, \infty) \to [0, \infty)$ as follows

$$A(r) = \sup\{|f(z)| : |z - r| = r\}$$

Prove that

$$A(1)^2 \leq A\big(2\big)\, A\big(\tfrac{2}{3}\big).$$