Basic Exam: Spring 2025

Test instructions:

- Write your UCLA ID number on the upper right corner of each page.
- Do not write your name anywhere on the exam.
- Your final score will be the sum of

  FIVE analysis problems (Problems 1–6) and

  FIVE linear algebra problems (Problems 7–12).

  However, to pass the exam you need to show mastery of both subjects.
- Indicate below which 10 problems you wish to have graded.
- Please staple your problems in numerical order.

1	2	3	4	5	6
7	8	9	10	11	12

Analysis problems

Problem 1. Let (X, d) be a compact metric space. Suppose that $f: X \to X$ is a continuous function such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove the following statements:

(a) f is an injective (one-to-one) function.

(b) f^{-1} is continuous on $f(X)$.

(c) f is surjective (onto).

Problem 2. Let $\{f_n\}$ and $\{g_n\}$ be sequences of Riemann integrable functions from $[0, 1]$ to $\mathbb{R}$ with the following properties:

(I) There exists a (finite) constant $C > 0$ independent of $n \in \mathbb{N}$ such that

$$\int_0^1 |g_n(x)| \, dx \leq C.$$

(II) f_n converges uniformly to f on $[0, 1]$.

(III) For all $n \in \mathbb{N}$, $\displaystyle \lim_{m \to \infty} \int_0^1 |f_n(x) g_m(x)| \, dx = 0.$

Prove that $\displaystyle \lim_{m \to \infty} \int_0^1 f(x) g_m(x) \, dx = 0.$

Problem 3. Let $u(x, y)$ be a real-valued C^2 (twice continuously differentiable) function on $S := \{ (x, y) \in \mathbb{R}^2 \mid y \geq 0 \}$. Suppose that

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad u(x, y) = u(x + 1, y) \tag{$*$}$$

for all $(x, y) \in S$. For $y \geq 0$, let $f(y) = \displaystyle \int_0^1 u(x, y)^2 \, dx.$

(a) Show that the function $f(y)$ is decreasing (not necessarily strictly) on $[0, \infty)$.

(b) Deduce that if two solutions $u(x, y)$ and $v(x, y)$ of $(*)$ are equal when $y = 0$, then they are equal for all $y \geq 0$.

Problem 4. Let (X, d) be a metric space. Prove that X is complete if and only if the following condition holds:

> for any nested sequence $E_1 \supset E_2 \supset \ldots$ of nonempty closed subsets of X whose diameters tend to 0, the intersection $\bigcap_{n \geq 1} E_n$ is nonempty.

Problem 5. Let (X, d) be a compact metric space, and let f_n be a sequence of continuous real-valued functions on X that converge (pointwise) to a continuous function f. Assume also that for all $x \in X$ and all n, $f_{n+1}(x) \leq f_n(x)$.

(a) Prove that the functions f_n converge to f uniformly.
Note: Don't just cite a theorem here. For credit, you must prove this.

(b) Show by a counterexample that the result in (a) does not hold if we drop the assumption that X is compact.

(c) Show by a counterexample that the result in (a) does not hold if we drop the assumption that the limit f is continuous.

Problem 6. Prove that there does not exist a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at every rational number but discontinuous at every irrational.
(Hint: Can you say anything general about the set of points where an arbitrary function $f: \mathbb{R} \to \mathbb{R}$ is continuous?)

Linear algebra problems

Problem 7. Let A be a symmetric $n \times n$ real matrix, and assume that A is positive semi-definite.

(a) Prove that for any $x \in \mathbb{R}^n$, the following Euclidean norm inequality holds:

$$\left\|(I - A)(I + A)^{-1}x\right\| \leq \|x\|.$$

(b) Prove that $x \in \ker(A)$ if and only if $(I - A)(I + A)^{-1}x = x$.

Problem 8. Suppose that A and B are 5×5 complex matrices that share the same eigenvectors. Prove that if the characteristic polynomial of A is x^5 and the minimal polynomial of B is $(x - 1)^2$, then $A^3 = 0$.

Problem 9. Fix $n \in \mathbb{N}$ and let $P_n \subseteq \mathbb{R}[x]$ be the vector space of all polynomials with real coefficients of degree at most n. Show that for each $a \in \mathbb{R}$, there is a unique polynomial $p_a \in P_n$ such that for all $q \in P_n$,

$$q(a) = \int_0^1 p_a(x)q(x)\,dx.$$

Problem 10. Let A be a 3×3 real matrix with $A^{\mathrm{T}} = -A$.

(a) Show that there is a 3×3 orthogonal matrix B and a real number c such that

$$BAB^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}.$$

(b) Show that the set $\{c, -c\}$ is uniquely determined by the original matrix A.

Problem 11. Let V be an n-dimensional vector space (over any field), and let T be a linear operator on V. Let $v \in V$ such that $T^k v = 0$ for some $k > 0$. Prove that $T^n v = 0$.

Problem 12. Let U, V, W be finite-dimensional inner product spaces and let $A\colon U \to V$ and $B\colon V \to W$ be linear functions such that $\operatorname{im}(A) = \ker(B)$. Show that the operator $AA^* + B^*B$ on V is invertible.