Geometry/Topology

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE
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1. ________________________ 5. ____________________________

2. ________________________ 6. ____________________________

3. ________________________ 7. ____________________________

4. ________________________ 8. ____________________________

Pass/fail recommend on this form.

Total score: ____________________

QUALIFYING EXAM
Geometry/Topology
March 2025

Attempt all ten problems. Each problem is worth 10 points. You must fully justify your answers.

1. Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost all $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely. (Here, "almost all" means the complement of some Lesbegue measure 0 subset.)

3. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with coordinates $(x, y) \in \mathbb{R}^2$ and let $p \in T^2$.
   (a) Compute the de Rham cohomology of the punctured torus $T^2 - \{p\}$.
   (b) Is the volume form $\omega = dx \wedge dy$ exact on $T^2 - \{p\}$?

4. Consider the 3-form on $\mathbb{R}^4$ given by
$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3.$$
   Let S^3 be the unit sphere in $\mathbb{R}^4$ and let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map.
   (a) Evaluate $\int_{S^3} \iota^*\alpha$.
   (b) Let γ be the 3-form on $\mathbb{R}^4 - \{0\}$ given by:
$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}$$
   for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

5. (a) Define what it means for a manifold M to be orientable. (You can give any one of the many equivalent definitions.)
   (b) Show that every nonorientable connected manifold M admits a connected, oriented double cover.

6. Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:
$$\chi(M) = \tfrac{1}{2}\chi(\partial M).$$

Continued on the next page.

7. Let M be a compact oriented n-manifold with cohomology group $H^1(M;\mathbb{R}) = 0$ and let T^n be the n-dimensional torus. For which integers k does there exist a smooth map $f\colon M \to T^n$ of degree k?

8. Exhibit a space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$, where $\mathbb{Z}/k\mathbb{Z}$ denotes the integers modulo k and $*$ denotes the free product. Exhibit another space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

9. Let L_x be the x-axis, L_y be the y-axis, and L_z be the z-axis of $\mathbb{R}^3$, and let $p = (1,1,1) \in \mathbb{R}^3$. Compute
$$\pi_1(\mathbb{R}^3 - L_x - L_y - L_z, p).$$

10. Let X be a topological space and $p \in X$. The *reduced suspension* ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the reduced homology groups of X and ΣX.