# **Numerical Analysis**

#### INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!** 

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: \_\_\_\_\_

DATE: \_\_\_\_\_

## EXAMINEES: DO NOT WRITE BELOW THIS LINE

1	5
2	6
3	7
4	8

# Pass/fail recommend on this form.

Total score:

### DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

**Problem** [1] (5 Pts.) Let  $f: \mathbb{R}^{n_0} \to \mathbb{R}$  be defined as

$$f = g_L \circ g_{L-1} \circ \cdots \circ g_2 \circ g_1$$

where  $g_i: \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i}$  for  $i = 1, \ldots, L$  and  $n_L = 1$ . Assume  $1 = n_L \leq n_{L-1} \leq \cdots \leq n_1 \leq n_0$ . Assume the computational cost of evaluating  $g_\ell$  and its Jacobian  $Dg_\ell$  is  $c_\ell$  for  $\ell = 1, \ldots, L$ . Given an  $x \in \mathbb{R}^n$ , describe an algorithm for computing  $\nabla f(x)$  that has complexity

$$\mathcal{O}\Big(\sum_{\ell=1}^{L-1} n_\ell n_{\ell-1} + \sum_{\ell=1}^L c_\ell\Big).$$

*Hint.* This problem is asking you to describe the implementation details of the chain rule.

Problem [2] (5 Pts.) Consider the approximation of

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y) \, dx dy \approx \sum_{i=1}^{n} f(x_i, y_i) w_i$$

with nodes  $\{(x_i, y_i)\}_{i=1}^n$  and weights  $\{w_i\}_{i=1}^n$ . Assume we wish to integrate *bivariate* polynomials of degree up to 5 exactly. Describe a construction using n = 9 nodes.

Hint. Gauss quadrature integrates univariate polynomials of degree up to 5 using 3 nodes.

**Problem [3] (5 Pts.)** Show that  $g(x) = \pi + 0.5 \sin(x/2)$  has a unique fixed point on the interval  $[0, 2\pi]$ . Give the fixed-point iteration algorithm for finding an approximation to the fixed point that is accurate to within  $10^{-2}$ . Estimate the number of iterations required to achieve this accuracy. Justify your answers.

**Problem** [4] (5 Pts.) The forward-difference formula can be expressed as

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

(a) Prove the above formula when f is a smooth function.

(b) Use extrapolation to derive an  $O(h^3)$  approximation formula for  $f'(x_0)$ .

**Problem** [5] (10 Pts.) Consider the autonomous ODE

$$y' = f(y), \qquad y(0) = y_0$$

for  $t \ge 0$ , where  $y \in \mathbb{R}^d$ . Assume  $f(\cdot)$  and  $y(\cdot)$  are smooth. Let  $\varphi_h$  for  $h \in \mathbb{R}$  (so h is not necessarily positive) be the exact flow map. Let

$$\Psi_h(y_n) \mapsto y_{n+1}$$

for  $h \in \mathbb{R}$  be a Runge–Kutta (RK) method of maximal order p, i.e.,

$$\Psi_h(y) = \varphi_h(y) + C(y)h^{p+1} + \mathcal{O}(h^{p+2}) \qquad \text{as } h \to 0$$

for any  $y \in \mathbb{R}^d$ , where C(y) is a smooth function of y. Define the *adjoint* method  $\Psi_h^*$  as

$$\Psi_h^*(y_n) = \Psi_{-h}^{-1}(y_n)$$

for  $h \in \mathbb{R}$ . Assume  $\Psi_h^*$  is well defined when |h| is small enough. Show that

$$\Psi_h^*(y) = \varphi_h(y) + (-1)^{p+1} C(y) h^{p+1} + \mathcal{O}(h^{p+2}) \qquad \text{as } h \to 0$$

for any  $y \in \mathbb{R}^d$ .

*Hint.* Let  $y_1 = \Psi_h^*(y_0)$ . Apply  $\Psi_{-h}$  and then  $\varphi_h$  to both sides.

Problem [6] (10 Pts.) Consider the equation

$$\frac{du}{dt} = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y},$$

to be solved for u(x, y, t), -1 < x, y < 1, t > 0, with u(x, y, 0) = h(x, y) given and smooth.

(a) What boundary conditions are needed on x = -1, x = 1, y = -1, y = 1 to make this well-posed? (b) Construct a convergent finite difference approximation for this initial-boundary value problem. Justify your answers. Problem [7] (10 Pts.) Consider the equation

$$\frac{du}{dt} = a \frac{\partial u}{\partial x} + b \frac{\partial^2 u}{\partial x^2}$$

for a, b positive constants, to be solved for u(x,t), 0 < x < 1, t > 0, periodic boundary conditions in x, u(x,0) = h(x) given and smooth.

Obtain a second order accurate explicit convergent approximation of the form

$$v(i, n + 1) = c(-2)v(i - 2, n) + c(-1)v(i - 1, n) + c(0)v(i, n) + c(1)v(i + 1, n) + c(2)v(i + 2, n)$$

for the constant coefficients, c(j), j = -2, -1, 0, 1, 2. Justify your answers.

**Problem [8] (10 pts)** Consider the following problem in a domain  $\Omega \subset \mathbb{R}^2$ , with  $\Gamma = \partial \Omega$ :

$$-\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma,$$

where  $\beta_i$  are constants.

(a) Choose an appropriate space of test functions V, and give a weak formulation of the problem.

(b) For any  $v \in V$ , show that

$$\int_{\Omega} \left( \beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2} \right) = 0.$$

(c) By analyzing the corresponding linear and bilinear forms, show that the weak formulation has a unique solution. Specify the necessary assumptions on f and  $\Omega$ .

(d) Set up a convergent, finite element approximation using  $P_1$  elements, and discuss the linear system thus obtained. Show that the linear system has a unique solution.

(e) Give the rate of convergence of your approximation.