Optimization/Numerical Linear Algebra

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________ 5. ______________________

2. ______________________ 6. ______________________

3. ______________________ 7. ______________________

4. ______________________ 8. ______________________

Pass/fail recommend on this form.

Total score: ____________________

Qualifying Exam, Spring 2025

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Consider the problem

$$\text{minimize} \quad x_1^2 x_2 + x_1^2 \quad \text{subject to} \quad x_1^2 - 2 - x_2 \leq 0 \text{ and } x_1^2 - 2 + x_2 \leq 0.$$

   (a) State the KKT conditions for this problem and find all points that satisfy them.

   (b) State the second order necessary and sufficient conditions and verify whether or not the points in part (a) satisfy them.

   (c) Write down the solution.

2. (10 points) Show that $z^T x = 0$ for all x satisfying $Ax = 0$ if and only if $z = A^T w$ for some w.
   Hint: Use the duality theorem of linear programming.

3. (10 points)

   a) Let $X = \{A \in \mathbb{R}^{n\times n} : A^T = A\}$ be the space of real symmetric matrices and let $f : X \to \mathbb{R}$ be the function mapping any $A \in X$ to its largest eigenvalue $f(A) = \lambda_{\max}(A)$. Given $A \in X$, let v be a normalized eigenvector of A corresponding to the eigenvalue $f(A)$. Show that $vv^T \in \partial f(A)$.

   b) Let X be a vector space and let $f : X \to (-\infty, +\infty]$ be proper (i.e., $\operatorname{dom} f := \{x \in X : f(x) < +\infty\} \neq \emptyset$). Show that if $\operatorname{dom} f$ is convex and $\operatorname{dom} f = \operatorname{dom} \partial f$, then f is convex.

4. (10 points) Consider the following description of the Conjugate Gradient method for full-rank symmetric positive-definite real-valued matrix $A \in \mathbb{R}^{n\times n}$ and the system $Ax = b$, where x_0 is chosen arbitrarily (you can assume at random).

$$\begin{aligned} p_0 &= r_0 = b - Ax_0 \\ \alpha_i &= \frac{\|r_i\|_2^2}{\langle p_i, Ap_i\rangle} \\ x_{i+1} &= x_i + \alpha_i p_i \\ r_{i+1} &= r_i - \alpha_i A p_i \\ \beta_i &= \frac{\|r_{i+1}\|_2^2}{\|r_i\|_2^2} \\ p_{i+1} &= r_{i+1} + \beta_i p_i \end{aligned}$$

a) Prove that for all j, r_j is the residual, that is $r_j = b - Ax_j$.

b) Prove that if the vectors $\{p_j\}$ are A-conjugate, then for all $i, j \leq n$, $i \neq j$ the vectors r_i and r_j are orthogonal. Conclude (explain why) that then (with perfect arithmetic and again assuming the vectors $\{p_j\}$ are A-conjugate), the solution x will be found by n iterations: $x_n = x$.

5. (10 points) Suppose A is diagonalizable with $A = S\Lambda S^{-1}$, where Λ has diagonal entries λ_j.

   a) Suppose you initialize the power method with a vector x_0 that is orthogonal to the dominant eigenvector v_1. Will the power method still converge? Prove your answer for the general case (i.e. do not just give an example).

   b) Prove that if $\lambda_1 = \lambda_2$ and $|\lambda_3| < |\lambda_1|$, then the method still offers convergence to an eigenvector of λ_1.

6. (10 points) Consider solving $Ax = b$ where $A = M - N \in \mathbb{R}^{n\times n}$ has unit-norm rows, using the splitting method with iteration given by the recurrence $Mx_k = b + Nx_{k-1}$. Assume both A and M are nonsingular.

   a) Prove that if $\|M^{-1}N\|^k \to 0$ then $x_k \to x$ for solution x as $k \to \infty$.

   b) Now assume the splitting method used is the Gauss-Jacobi method (with $M = D$, where D is the diagonal component of A). Suppose the stopping criterion is such that the algorithm terminates when $\|x_k - x_{k-1}\| \leq \varepsilon$. Show that after the algorithm terminates at iteration T, the residual norm is bounded: $\|Ax_T - b\|_2 \leq C$, and express C as a function of n and ε.

7. (10 points)

(a) Let $A \in \mathbb{R}^{m\times n}$. Let $\|\cdot\|_2$ denote the spectral norm of a matrix, which is the largest singular value.

(i) Let $B \in \mathbb{R}^{m\times n}$ be another matrix. Show that

$$\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2^2 \leqslant \|A\|_2^2 + \|B\|_2^2.$$

(ii) Give an example where equality holds in, and one where it does not.

(b) Consider the least-squares problem

$$\min_x \|Ax - b\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$ with $m \geqslant n$ and $\operatorname{rank}(A) = n$. Here $\|\cdot\|_2$ for a vector denotes the Euclidean norm.

(i) Explain how to solve the problem using the QR factorization of A and give an explicit algebraic expression for the solution x_*.

(ii) When is $x_* = 0$ the solution? Give a characterization in terms of A and b.

(iii) Let $A_1, A_2 \in \mathbb{R}^{m\times n}$ with $m > 2n$ and $\operatorname{rank}(A_1) = \operatorname{rank}(A_2) = n$, and suppose that $A_1^T A_2 = 0$. Let $x_{1,*}$ and $x_{2,*}$ be defined as the solutions of the least-squares problems

$$x_{1,*} = \operatorname{argmin}_x \|A_1 x - b\|_2\,, \quad x_{2,*} = \operatorname{argmin}_x \|A_2 x - b\|_2\,.$$

Express the solution y_* of the least-squares problem

$$y_* = \operatorname{argmin}_y \|By - b\|_2, \quad \text{where} \quad B = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \in \mathbb{R}^{m\times 2n}$$

in terms of $x_{1,*}$ and $x_{2,*}$.

8. (10 points) Let $A \in \mathbb{R}^{m\times n}$. The Frobenius norm is defined by $\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2}$.

   (a) Show that the Frobenius norm is orthogonally invariant, that is, $\|QAV\|_F = \|A\|_F$ for any orthogonal matrices $Q \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$.

   (b) Show that $\|A\|_F^2 = \sum_{i=1}^{\min(m,n)} (\sigma_i(A))^2$, where $\sigma_i(A)$ is the i-th largest singular value of A.

   (c) Let $B \in \mathbb{R}^{n\times \ell}$. Prove that $\|AB\|_F \leqslant \|A\|_F \|B\|_2$, where $\|B\|_2 = \sigma_1(B)$ is the spectral norm of B.

   Hint: Use Courant-Fischer to first show $\sigma_i(AB) \leqslant \sigma_i(A)\|B\|_2$ as follows

$$\sigma_i(A) = \max_{QQ^*=I_i} \min_{\|x\|=1} \|x^* QA\|_2. \tag{2}$$

9. (10 points)

(a) Let $A \in \mathbb{R}^{m \times n}, m \geqslant n$ and $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0$ be its singular values. Prove that for $k = 1, 2, \ldots, n$,

$$\sum_{i=1}^{k} \sigma_i(A) = \max_{Q^T Q = I_k, W^T W = I_k} \operatorname{trace}\left(Q^T A W\right), \tag{3}$$

where $Q \in \mathbb{R}^{m \times k}, W \in \mathbb{R}^{n \times k}$ are orthonormal.

(b) If $Ax = b$ and $(A + \delta A)(x + \delta x) = b$ show the inequality below and interpret the result

$$\frac{\|\delta x\|}{\|x + \delta x\|} \leqslant \|A\| \left\|A^{-1}\right\| \frac{\|\delta A\|}{\|A\|} \tag{4}$$

Hint: You can use without proof the fact that $\sigma_i(AB) \leqslant \sigma_i(A) \|B\|_2$.