ALGEBRA QUALIFYING EXAM

MARCH 25, 2025

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

Complete 8 of the 10 problems. Clearly indicate, by circling the numbers of the problems below, which 8 problems should be graded. Throughout the exam, justify your answers.

$$\boxed{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10}$$

No books, notes, phones, or other printed or electronic materials can be used on the exam.

Please staple your problems in the order they are listed in the exam.

Problem 1. Let p be an odd prime number and let K be the splitting field of the polynomial $x^p - 1$ over $\mathbb{Q}$. Show that K contains a unique subfield F such that $[F : \mathbb{Q}] = 2$ and determine, as property of p, whether F is a real or a complex quadratic extension of $\mathbb{Q}$.

Problem 2. Let R be a UFD. Show that any non-zero prime ideal of $R[x]$ which contains no non-zero elements of R is principal.

Problem 3.
1. Show that the alternating group A_4 has a unique irreducible representation of degree 3 over $\mathbb{C}$. State clearly any general theorems that you use.
2. Compute the character of this representation.

Problem 4. Let R be an integral domain with quotient field F. Show that if F is finitely generated as an R-module, then $F = R$. State fully any theorems that you use.

Problem 5. Let G be a p-group for p a prime. Let F be a field of characteristic p. Show that the only irreducible representation of G in finite dimensional F vector spaces is the trivial representation.

Problem 6. Let R be a ring and let F be the forgetful functor from the category of left R-modules to the category of abelian groups that takes an R-module M to M considered as an abelian group. Determine, with full proofs, left and right adjoint functors for F.

Problem 7. Prove that the group $S_5 \times S_5$ is generated by two elements.

Problem 8. Prove that the ring of all algebraic integers is not a UFD.

Problem 9. Let A be an $n \times n$ matrix over a field F such that the characteristic polynomial of A is its minimal polynomial. Let B be an $n \times n$ matrix over F such that $AB = BA$. Prove that $B = f(A)$, where f is a polynomial over F.

Problem 10. Let F be a field and let A be a simple F-algebra of dimension n^2. Prove that $A \simeq M_n(F)$ if and only if A has a left ideal of dimension n over F.