Qualifying Exam, Spring 2025

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

Problem [1] (5 Pts.) Let $f\colon \mathbb{R}^{n_0} \to \mathbb{R}$ be defined as

$$f = g_L \circ g_{L-1} \circ \cdots \circ g_2 \circ g_1$$

where $g_i\colon \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i}$ for $i = 1, \dots, L$ and $n_L = 1$. Assume $1 = n_L \le n_{L-1} \le \cdots \le n_1 \le n_0$. Assume the computational cost of evaluating g_ℓ and its Jacobian Dg_ℓ is c_ℓ for $\ell = 1, \dots, L$. Given an $x \in \mathbb{R}^n$, describe an algorithm for computing $\nabla f(x)$ that has complexity

$$\mathcal{O}\Big(\sum_{\ell=1}^{L-1} n_\ell n_{\ell-1} + \sum_{\ell=1}^{L} c_\ell\Big).$$

Hint. This problem is asking you to describe the implementation details of the chain rule.

Problem [2] (5 Pts.) Consider the approximation of

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy \approx \sum_{i=1}^{n} f(x_i,y_i)w_i$$

with nodes $\{(x_i,y_i)\}_{i=1}^n$ and weights $\{w_i\}_{i=1}^n$. Assume we wish to integrate *bivariate* polynomials of degree up to 5 exactly. Describe a construction using $n = 9$ nodes.

Hint. Gauss quadrature integrates *univariate* polynomials of degree up to 5 using 3 nodes.

Problem [3] (5 Pts.) Show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on the interval $[0, 2\pi]$. Give the fixed-point iteration algorithm for finding an approximation to the fixed point that is accurate to within 10^{-2}. Estimate the number of iterations required to achieve this accuracy. Justify your answers.

Problem [4] (5 Pts.) The forward-difference formula can be expressed as

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

(a) Prove the above formula when f is a smooth function.

(b) Use extrapolation to derive an $O(h^3)$ approximation formula for $f'(x_0)$.

Problem [5] (10 Pts.) Consider the autonomous ODE

$$y' = f(y), \qquad y(0) = y_0$$

for $t \geq 0$, where $y \in \mathbb{R}^d$. Assume $f(\cdot)$ and $y(\cdot)$ are smooth. Let φ_h for $h \in \mathbb{R}$ (so h is not necessarily positive) be the exact flow map. Let

$$\Psi_h(y_n) \mapsto y_{n+1}$$

for $h \in \mathbb{R}$ be a Runge–Kutta (RK) method of maximal order p, i.e.,

$$\Psi_h(y) = \varphi_h(y) + C(y)h^{p+1} + \mathcal{O}(h^{p+2}) \qquad \text{as } h \to 0$$

for any $y \in \mathbb{R}^d$, where $C(y)$ is a smooth function of y. Define the *adjoint* method Ψ_h^* as

$$\Psi_h^*(y_n) = \Psi_{-h}^{-1}(y_n)$$

for $h \in \mathbb{R}$. Assume Ψ_h^* is well defined when $|h|$ is small enough. Show that

$$\Psi_h^*(y) = \varphi_h(y) + (-1)^{p+1}C(y)h^{p+1} + \mathcal{O}(h^{p+2}) \qquad \text{as } h \to 0$$

for any $y \in \mathbb{R}^d$.

Hint. Let $y_1 = \Psi_h^*(y_0)$. Apply Ψ_{-h} and then φ_h to both sides.

Problem [6] (10 Pts.) Consider the equation

$$\frac{du}{dt} = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y},$$

to be solved for $u(x, y, t)$, $-1 < x,\ y < 1$, $t > 0$, with $u(x, y, 0) = h(x, y)$ given and smooth.

(a) What boundary conditions are needed on $x = -1$, $x = 1$, $y = -1$, $y = 1$ to make this well-posed?

(b) Construct a convergent finite difference approximation for this initial-boundary value problem.

Justify your answers.

Problem [7] (10 Pts.) Consider the equation

$$\frac{du}{dt} = a\frac{\partial u}{\partial x} + b\frac{\partial^2 u}{\partial x^2},$$

for a, b positive constants, to be solved for $u(x,t)$, $0 < x < 1$, $t > 0$, periodic boundary conditions in x, $u(x,0) = h(x)$ given and smooth.

Obtain a second order accurate explicit convergent approximation of the form

$$v(i,n+1) = c(-2)v(i-2,n) + c(-1)v(i-1,n) + c(0)v(i,n) + c(1)v(i+1,n) + c(2)v(i+2,n)$$

for the constant coefficients, $c(j)$, $j = -2, -1, 0, 1, 2$. Justify your answers.

Problem [8] (10 pts) Consider the following problem in a domain $\Omega \subset \mathbb{R}^2$, with $\Gamma = \partial\Omega$:

$$-\triangle u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma,$$

where β_i are constants.

(a) Choose an appropriate space of test functions V and give a weak formulation of the problem.

(b) For any $v \in V$, show that

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}\right) v(x)dx = 0.$$

(c) By analyzing the corresponding linear and bilinear forms, show that the weak formulation has a unique solution. Specify the necessary assumptions on f and Ω.

(d) Set up a convergent, finite element approximation using P_1 elements, and discuss the linear system thus obtained. Show that the linear system has a unique solution.

(e) Give the rate of convergence of your approximation.