# ADE Exam, Fall 2025 Department of Mathematics, UCLA

1. [10 points] (a) Consider the following system of ODEs for the pair $(x(t), v(t))$ of real-valued functions:
$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -v - \frac{1}{2}x\,(x^2 - 1), \tag{1}$$
for $t > 0$, with initial conditions $x(0) = x_0$ and $v(0) = v_0$. Find all stationary points and sketch the local phase portraits.

   (b) Consider the following system of ODEs for the pair $(x(t), y(t))$ of real-valued functions:
$$\dot{x} = -y + \alpha x(x^2 + y^2),$$
$$\dot{y} = x + \alpha y(x^2 + y^2),$$
where $\alpha \in \mathbb{R}$ is a constant. Determine, with appropriate arguments, the *nonlinear* stability of the equilibrium point at the origin. Also draw the phase portraits for this system for all qualitatively different values of $\alpha$.

2. [10 points] Using a Frobenius-series approach near $t = 0$, find two independent solutions of the differential equation
$$t^2\frac{d^2y}{dt^2} + (t^2 + t)\frac{dy}{dt} - \frac{1}{4}y = 0, \quad 0 < t < \infty.$$
Derive the recurrence relations for the coefficients of the series expansions.

3. [10 points] Consider the eigenvalue problem on $[0, \pi/2]$.
$$u_{xx} = -\lambda u, \quad u(0) = 0, \quad u_x(\pi/2) = 0.$$
   (a) Find all the eigenfunctions and eigenvalues for this problem.

   (b) Use the answer from part (a) to write down a closed form solution of the heat equation
$$u_t - u_{xx} = 0, \quad u(0) = 0, \quad u_x(\pi/2) = 0,$$
with initial condition
$$u(x, 0) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin((2n-1)x).$$

4. [10 points] Solve the forward-time negative-flux Burgers equation $u_t - uu_x = 0$ with the initial condition
$$u(x, 0) = 1; \quad 0 < x < 1;$$
$$u(x, 0) = 0; \quad \text{otherwise}$$
on the time interval $0 < t < 1$. Please find the solution that satisfies the entropy condition. Hint: draw the characteristic diagram going forward in time. Assess which jump leads to a rarefaction vs. a shock.

5. [10 points - 2pts each] Which of the following is a bounded operator on $L^2(R^n)$? Either prove boundedness or provide a counterexample.

   (a) the gradient operator
   (b) $(I + c\Delta^2)^{-1}$, $c > 0$,
   (c) $I - c\Delta$, $c > 0$,
   (d) $F(u) = u^2$,
   (e) $F(u) = u$.

6. [10 points] Consider the two-dimensional Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0 \qquad \text{for } r^2 > a^2\,, \\ u &= \sin(\theta) \quad \text{for } r^2 = a^2\,, \end{aligned} \tag{2}$$

   with $u(r, \theta)$ bounded as $r \to \infty$.

   Show that

$$u(r, \theta) = (r^2 - a^2) \int_0^{2\pi} \frac{\sin(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi} \tag{3}$$

   for $r > a$.

7. [10 points] Consider the initial-value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t)\,, \quad -\infty < x < \infty\,,\ t > 0\,, \\ u(x,0) &= \phi(x)\,, \qquad -\infty < x < \infty\,, \\ u_t(x,0) &= \psi(x)\,, \qquad -\infty < x < \infty\,. \end{aligned} \tag{4}$$

   Using Green's theorem, show that a solution $u(x,t)$ of (4) satisfies

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds + \frac{1}{2c}\iint_D f(y,\tau)\, dy\, d\tau\,, \tag{5}$$

   where $D$ is the domain of dependence that is associated with $(x,t)$ (i.e., the triangle in the $xt$-plane with top point $(x,t)$ and base points $(x - ct, 0)$ and $(x + ct, 0)$).

   Briefly indicate a physical interpretation of the solution form (5).

   *[Note: You may assume any desirable continuity and differentiability properties of the functions $f$, $\phi$, and $\psi$ provided that you state them explicitly and precisely.]*

8. [10 points] Using the method of characteristics, carefully derive the solution $u(x,y,z)$ of

$$xu_x + 2yu_y + u_z = 3u\,,\ u(x,y,0) = \sin(x+y)\,. \tag{6}$$