ALGEBRA QUALIFYING: FALL 2025

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

On the front of your paper indicate which 8 problems you wish to have graded.

Please staple your problems in the order they are listed in the exam.

1	2	3	4
5	6	7	8
9	10		

Problem 1. Let R be a commutative ring with identity. Assume that $I \cap J = IJ$ for every two ideals I and J in R. Prove that every prime ideal of R is maximal.

Solution 1:

Problem 2. Let

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

be an exact sequence of torsion-free modules over an integral domain R. Let Q be the quotient field of R. Let $s\colon B \otimes_R Q \to A \otimes_R Q$ and $t\colon C \otimes_R Q \to B \otimes_R Q$ be Q-vector space splittings of $\iota \otimes \mathrm{id}$ and $\pi \otimes \mathrm{id}$ respectively such that $\iota \circ s + t \circ \pi = 1$. Show that

$$s(B)/A \cong C/\pi(t(C) \cap B)$$

as R-modules.

Solution 2:

Problem 3. Let G be a finite group. Assume that $h^2g^2 = g^2h^2$ for every two elements h and g in G. Prove that G is solvable.

Hint: Consider the subgroup of G generated by the squares of all elements.

Solution 3:

Problem 4. Let p be a prime number. Consider the group of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\} \leq \mathrm{GL}_2(\mathbb{F}_p).$$

Find the degrees of the characters of G, with multiplicity.

Solution 4:

Problem 5. Let $\mathcal{C}$ be a category that admits pullbacks and products of two objects of $\mathcal{C}$. Given morphisms $f, g \colon A \to B$ in $\mathcal{C}$, construct an equalizer $\iota \colon E \to A$ of f and g in $\mathcal{C}$. Show that $f \circ \iota = g \circ \iota$, but do not show that ι actually has the universal property of the equalizer.

Solution 5:

Problem 6. Let F be a field and $R = F[x_1, x_2]$. Give F the R-module structure on which $f \in R$ acts as multiplication by $f(0,0)$.

(i) Find a resolution of F by projective R-modules of minimal length.
(ii) Compute $\mathrm{Tor}^i_R(F, F[x_1])$, where $F[x_1]$ is given the R-module structure on which $f \in R$ acts as $f(x_1, 0)$.

Solution 6:

Problem 7. Let R be an artinian ring, and let J be the intersection of its left maximal ideals. Show that every element of J is nilpotent.

Solution 7:

Problem 8. Let R be a commutative ring with identity. Let M be a finitely generated R-module. Let $f\colon M \to M$ be a surjective homomorphism of R-modules. Prove that f is injective.

Solution 8:

Problem 9. Let R be a PID. Let $M_1 \supseteq M_2 \supseteq M_3$ be three R-modules with $M_1 \simeq M_3 \simeq R^n$ as R-modules. Prove that $M_2 \simeq R^n$. Show a counter-example of this statement when R is not a PID.

Solution 9:

Problem 10. Let K be an extension of $\mathbb{Q}$ contained in $\mathbb{C}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order 4. Prove that i, a square root of -1, is not contained in K.

Solution 10: