Analysis Qualifying Exam, Fall 2025

ID No.:

Instructions and Rubric

- There are 12 problems: 6 on real analysis, 6 on complex analysis.
- Attempt at most five questions on real analysis and five questions on complex analysis. If you submit answers to more questions than this, please indicate clearly which questions should be graded.
- All questions will be graded out of 10 points. Questions with several parts show the breakdown of points in square brackets.
- In case of partial progress on a problem, details will usually earn more points if they are explained as part of a solution outline for the whole problem.

Conventions

- All Banach spaces are over the reals unless indicated otherwise.
- All functions are real-valued unless indicated otherwise.
- L^p always refers to Lebesgue measure unless a different measure is indicated.
- $\mathbb{N}, \mathbb{R}, \mathbb{C}$ are the sets of positive integers and real and complex numbers, respectively, and D is the open unit disk in $\mathbb{C}$.

Real Analysis

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x) = e^{-\sqrt{\|x\|}}, \quad x \in \mathbb{R}^2$$

where $\|x\|$ is the Euclidean norm of x in $\mathbb{R}^2$. Prove

$$\lim_{\|a\| \to +\infty} \frac{(f * f)(a)}{f(a)} = 2 \int_{\mathbb{R}^2} e^{-\sqrt{\|x\|}} dx, \quad x \in \mathbb{R}^2, \quad a \in \mathbb{R}^2$$

Here the $*$ operation stands for the convolution.

2. Let H be a separable Hilbert space and $T : H \to H$ be a compact operator (i.e. T maps bounded subsets of H to relatively compact subsets of H).

   (a) Prove that for every $\varepsilon > 0$, there exists a finite-rank operator T_ε (i.e. an operator whose range is contained in a finite-dimensional subspace of H) such that

   $$\|T - T_\varepsilon\|_{\mathrm{op}} \leq \varepsilon.$$

   Hint: One may take $T_\varepsilon = P \circ T$, where P is an orthogonal projection onto a finite-dimensional subspace.

   (b) Show that for any orthonormal basis $\{x_n\}_{n=1}^{\infty}$ of H,

   $$\lim_{n \to \infty} \|T x_n\| = 0.$$

3. Find two *explicit, non-zero* real-valued sequences $x_n, y_n \in \ell^2(\mathbb{Z})$ such that

$$\sum_{m \in \mathbb{Z}} x_{n-m} y_m = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

4. Let M denote the Hardy-Littlewood maximal function defined via

$$(Mf) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.$$

(a) Show that for any $f \in L^1(\mathbb{R}^n)$ and any $\lambda > 0$ we have

$$|\{x \in \mathbb{R}^d : [Mf](x) > \lambda\}| \leq \frac{C}{\lambda} \int_{|f| > \frac{\lambda}{2}} |f(y)| \, dy$$

for some constant $C > 0$.

(b) Deduce that if f is supported on $B(0,1)$ and $|f| \log[2 + |f|] \in L^1(B(0,1))$, then $Mf \in L^1(B(0,1))$.

5. We define a sequence of functions on $\mathbb{R}$ by

$$\psi_n(x) = \left[\frac{d}{dx} - 2\pi x\right]^n e^{-\pi x^2}$$

where $n \geq 0$. Show that $\psi_n(x)$ form an orthogonal sequence of eigenfunctions for the Fourier transform on $L^2(\mathbb{R})$. Our convention for the Fourier transform is

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) \, dx.$$

6. Fix $f \in L^2(\mathbb{R})$. Show that

$$\lim_{h \to \infty} \int f(x - y) h^2 y e^{-h|y|} \, dy$$

exists and is equal to zero at almost every $x \in \mathbb{R}$.

Complex Analysis

1. Construct an entire function whose zeros are precisely at the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, such that the zero at $k \in \mathbb{N}$ has multiplicity $k!$, for each $k \in \mathbb{N}$.

2. Let

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\},$$

and let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic function such that

$$f(i) = i, \qquad f(2i) = \tfrac{15}{8} i.$$

Prove that

$$\lim_{n\to\infty} f^{\circ n}(2i) = i,$$

where $f^{\circ n}$ denotes the n-th iterate of f.

3. Calculate
$$\int_0^\infty \frac{\log x}{x^2+4}\,dx$$

4. Find an explicit constant $C > 0$ so that
$$|\nabla u(x,y)| \le C(1-x^2-y^2)^{-1}\|u\|_{L^\infty}$$
for every bounded harmonic function $u : \mathbb{D} \to \mathbb{R}$. Prove that your choice of C works.

5. Let f and g be entire functions such that
$$f^3 + g^3 = 1 \quad \text{on } \mathbb{C}.$$
Show that f and g are constant.

   Hint: Factorize the left-hand side.

6. (a) Construct an explicit conformal mapping f of the slit unit disk
$$\Omega = \left\{z \in \mathbb{D} : z \notin \left[-\tfrac{1}{2}, 1\right)\right\}$$
onto the unit disk $\mathbb{D}$ with the additional property that $f(-\frac{3}{4}) = 0$ and $f'(-\frac{3}{4}) > 0$. It is preferred that you write f as a composition of simpler maps.

   (b) Express $f(\overline{z})$ in terms of $f(z)$.