Basic Exam: Fall 2025

Test instructions:

- Write your UCLA ID number on the upper right corner of each page.
- Do not write your name anywhere on the exam.
- Your final score will be the sum of

  FIVE analysis problems (Problems 1-6) and

  FIVE linear algebra problems (Problems 7-12).

  However, to pass the exam you need to show mastery of both subjects.
- Indicate below which 10 problems you wish to have graded.
- Please staple your problems in numerical order.

1	2	3	4	5	6
7	8	9	10	11	12

Analysis Problems

(1) Let $\{x_n\}_{n\ge 0}$ be a sequence of reals given recursively by $x_{n+1} = x_n - x_n^2$ with $x_0 = 1/2$. Prove that $x_n \to 0$. Then show that $\{nx_n\}_{n\ge 0}$ has a limit and

$$\lim_{n\to\infty} nx_n = 1$$

Hint: For the second part, consider the increments of the sequence $\{1/x_n\}_{n\ge 0}$.

(2) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is everywhere differentiable albeit with f' possibly not continuous. Prove that the derivative f' maps intervals into intervals (possibly unbounded); more precisely, that $f'(I)$ is either an interval or a single point whenever $I \subseteq \mathbb{R}$ is a bounded interval. *Hint:* Consider the function $x \mapsto f(x) - tx$ for various t.

(3) Let $f\colon [0,\infty) \to [0,\infty)$ be continuous on $[0,\infty)$ and continuously differentiable on $(0,\infty)$ with $\int_0^\infty |f'(x)|\mathrm{d}x < \infty$. Prove that the limit

$$\lim_{n\to\infty}\left(\sum_{k=1}^n f(k) - \int_0^n f(x)\mathrm{d}x\right)$$

exists.

(4) Prove that there exists a continuous function $f\colon [0,1] \to [0,1]$ such that

$$f(x) = \mathrm{e}^{-x}\int_0^1 \mathrm{e}^{x[f(t)-1]}\mathrm{d}t$$

holds for all $x \in [0,1]$.

(5) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and, for all integers $n \ge 1$ and all $x \in \mathbb{R}$, define

$$f_n(x) := f(nx + \sqrt{n}).$$

Assuming that the family $\{f_n\}_{n=1}^\infty$ is equicontinuous at every point of $[-1,1]$, prove that f is a constant function.

(6) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an everywhere-defined continuously-differentiable function such that $\|f(x)\|_2 \to \infty$ as $\|x\|_2 \to \infty$ and such that the Jacobian of f is non-vanishing everywhere. Prove that f maps $\mathbb{R}^2$ onto $\mathbb{R}^2$ (i.e., is a surjection).

Linear Algebra Problems

(7) Solve the system of ordinary differential equations

$$\begin{aligned} x'(t) &= 3x(t) - y(t) \\ y'(t) &= -2x(t) + 4y(t) \end{aligned}$$

for functions $t \mapsto x(t)$ and $t \mapsto y(t)$ on $\mathbb{R}_+$ subject to the initial data $x(0) = 1$ and $y(0) = 0$.

(8) Let V be a vector space over a field F and write $V^{\#} := \{\ell\colon V \to F \text{ linear}\}$ for its linear dual. Define $\alpha\colon V \to (V^{\#})^{\#}$ by $(\alpha(v))(\ell) := \ell(v)$, for all $v \in V$ and $\ell \in V^{\#}$. Show that if α is surjective (onto) then V is finite-dimensional. *Hint:* Consider the dual vectors of a basis of V and construct an interesting vector in $(V^{\#})^{\#}$.

(9) Let V be a finite dimensional inner product space over $\mathbb{C}$ with the associated norm denoted as $\|\cdot\|$. Let $L\colon V \to V$ be a self-adjoint linear operator, $\gamma \in \mathbb{R}$ and $\epsilon > 0$. Prove the following statement: If there exists a vector $x \in V$ such that $\|x\| = 1$ and

$$\|L(x) - \gamma x\| < \epsilon,$$

then L has an eigenvalue λ which satisfies $|\gamma - \lambda| < \epsilon$.

(10) Let $\mathrm{Mat}_{n\times n}(\mathbb{R})$ be the set all real $n \times n$ matrices. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a fixed matrix and define the function $f_A\colon \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ by

$$f_A(X) := A^{\mathrm{T}} X A$$

Do as follows:

(a) Prove that

$$\langle X, Y\rangle := \mathrm{Tr}(X^{\mathrm{T}} Y)$$

defines a scalar product and f_A defines a linear operator on $\mathrm{Mat}_{n\times n}(\mathbb{R})$.

(b) Endowing $\mathrm{Mat}_{n\times n}(\mathbb{R})$ with the above scalar product, prove that the adjoint of f_A is $f_{A^{\mathrm{T}}}$, i.e.,

$$(f_A)^* = f_{A^{\mathrm{T}}}$$

(11) Let A be a real $p \times q$ matrix of rank q. Prove that $A^{\mathrm{T}} A$ is invertible.

(12) Let A be a complex valued matrix and assume it is invertible. Prove that A^k is diagonalizable for any positive integer $k \geq 2$ if and only if A is diagonalizable.