

Qualifying Exam: Geometry/Topology Fall 2025

Instructions: In order to pass the exam you must receive at least 60/100 AND (receive at least 8/10 on six problems OR (receive at least 8/10 on five problems AND receive at least 5/10 on two other problems)).

Problem 1: Consider the space of all straight lines in $\mathbb{R}^2$ (not only those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Problem 2: Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Problem 3: Consider the vector field $X(z) = z^{2025} + 2025z^{2024} + 2025$ on $\mathbb{C} \simeq \mathbb{R}^2$, i.e., for $z \in \mathbb{C}$ identify $T_z\mathbb{C} = \mathbb{C}$, and write $X(z) = z^{2025} + 2025z^{2024} + 2025 \in T_z\mathbb{C}$. Compute the sum of the indices of X over all the zeros of X .

Problem 4: Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with coordinates $(x, y) \in \mathbb{R}^2$ and let $p \in T^2$.

- (a) Compute the de Rham cohomology of the punctured torus $T^2 - \{p\}$.
- (b) Is the volume form $\omega = dx \wedge dy$ exact on $T^2 - \{p\}$?

Problem 5: Consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = x^1 dx^2 \wedge dx^3 \wedge dx^4 - x^2 dx^1 \wedge dx^3 \wedge dx^4 + x^3 dx^1 \wedge dx^2 \wedge dx^4 - x^4 dx^1 \wedge dx^2 \wedge dx^3.$$

Let $S^3 \subset \mathbb{R}^4$ be the unit sphere and $\iota : S^3 \rightarrow \mathbb{R}^4$ the inclusion map.

- (a) Evaluate $\int_{S^3} \iota^* \alpha$.
- (b) Let γ be the 3-form on $\mathbb{R}^4 - \{0\}$ given by:

$$\gamma = \frac{\alpha}{((x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2)^k}$$

for $k \in \mathbb{R}$. Determine the values of k for which γ is closed and those for which it is exact.

Problem 6: Let M be a compact odd-dimensional manifold with nonempty boundary ∂M . Show that the Euler characteristics of M and ∂M are related by: $\chi(M) = \frac{1}{2}\chi(\partial M)$.

Problem 7: Exhibit a topological space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$, where $\mathbb{Z}/k\mathbb{Z}$ denotes the integers modulo k and $*$ denotes the free product. Exhibit another space whose fundamental group is isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Problem 8: Let L_x be the x -axis, L_y be the y -axis, and L_z be the z -axis of $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 - L_x - L_y - L_z, p_0)$, $p_0 \in \mathbb{R}^3 - L_x - L_y - L_z$.

Problem 9: Let X be a topological space and $p \in X$. The reduced suspension ΣX of X is the space obtained from $X \times [0, 1]$ by contracting $(X \times \{0, 1\}) \cup (\{p\} \times [0, 1])$ to a point. Describe the relation between the homology groups of X and ΣX .

Problem 10: Let M be a compact oriented n -manifold such that $H_1(M, \mathbb{Q}) = 0$ and let T^n be the n -dimensional torus. For which integers k does there exist a map $f : M \rightarrow T^n$ of degree k ?