Qualifying Examination
LOGIC
Fall 2025

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

Problem 1. Give an example of a pair of countable connected graphs that are elementarily equivalent but not isomorphic.

Problem 2. Show that a theory T has quantifier elimination if and only if for every model $\mathbf{M} \models T$ and every substructure $\mathbf{A} \subseteq \mathbf{M}$, the theory $T \cup \mathsf{Diag}(\mathbf{A})$ is complete. Here $\mathsf{Diag}(\mathbf{A})$ is the atomic diagram of $\mathbf{A}$. If you use any criterion for quantifier elimination other than its definition, you must prove it.

Problem 3. Does there exist a consistent r.e. theory T extending PA that proves its own **in**consistency, i.e., $T \vdash \neg\mathsf{Con}(T)$?

Problem 4. Let T be an r.e. extension of PA such that $T \vdash \mathsf{Provable}_T(\ulcorner\varphi\urcorner) \to \varphi$ for every sentence φ (i.e., T proves that "provable sentences are true"). Show that T is inconsistent.

Problem 5. Give an example of a theory T that has models of cardinality $\aleph_0$ and $2^{\aleph_0}$ but no models $\mathbf{M}$ with $\aleph_0 < |M| < 2^{\aleph_0}$. **Hint.** You may use, without proof, the fact that there exists an almost disjoint family $\mathcal{F}$ of subsets of ω of cardinality $|\mathcal{F}| = 2^{\aleph_0}$. (A family $\mathcal{F}$ of sets is *almost disjoint* if the pairwise intersections of the sets in $\mathcal{F}$ are finite.)

Problem 6. Assume $V = L$ and prove the following guessing principle: There is a sequence $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ such that each $\mathcal{A}_\alpha$ is a countable family of subsets of $\omega \times \alpha$, and such that for every set $A \subseteq \omega \times \omega_1$, the set $\{\alpha : A \cap (\omega \times \alpha) \in \mathcal{A}_\alpha\}$ contains a club.

Problem 7. Prove that the guessing principle in Problem 6 implies $\diamond_{\omega_1}$.

Problem 8. Let $\mathcal{L}$ be the language of arithmetic with an additional constant symbol $\tilde{c}$. Call an $\mathcal{L}$-sentence *simple* if it has the form $\tilde{c} = S(\cdots S(0)\cdots)$. Let $\theta \mapsto \ulcorner\theta\urcorner$ be a Gödel numbering of the formulas of $\mathcal{L}$, and let φ_n denote the formula such that $\ulcorner\varphi_n\urcorner = n$. For every $\mathcal{L}$-sentence θ, let C_θ be the set of all natural numbers n such that $\mathsf{PA} \cup \{\theta\} \vdash \varphi_n$. Recall that W_e denotes the domain of the e-th partial unary recursive function.

(8a) Prove that there are total recursive functions $e \mapsto \ulcorner\theta_e\urcorner$ and $i \mapsto n_i$ such that for every e and i, θ_e is a simple $\mathcal{L}$-sentence and we have $i \in W_e$ if and only if $\mathsf{PA} \cup \{\theta_e\} \vdash \varphi_{n_i}$.

(8b) Let A be the set of all indices e for which there exists a simple $\mathcal{L}$-sentence θ such that $W_e \subseteq C_\theta$. Classify A in the arithmetical hierarchy.