Qualifying Exam, Fall 2025

Numerical Analysis

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 7 problems. Problems 1-2 are worth 5 points and problems 3-7 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems [1-3] and [5-7] to obtain a passing score.

Problem [1] (5 Pts.) Let $l(x)$ be a straight line which is the best approximation of $\sin x$ in the sense of the method of least squares over the interval $[-\pi/2,\ \pi/2]$. Show that the residual $d(x) = \sin x - l(x)$ is orthogonal to any second degree polynomial. The scalar product is given by

$$(f, g) = \int_{-\pi/2}^{\pi/2} f(x)g(x)dx.$$

Problem [2] (5 Pts.) Obtain a generalized trapezoidal rule of the form

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1) + ph^2(f_0' - f_1'),$$

where $x_i = x_0 + ih$ and $f_i = f(x_i)$, $f_i' = f'(x_i)$. Find the constant p and the error term. Deduce the composite rule for integrating

$$\int_a^b f(x)dx,\ a = x_0 < x_1 < ... < x_N = b.$$

Problem [3] (10 Pts.) Consider applying Newton's iteration to find a root of multiplicity 2. Assume $\alpha \in \mathbb{R}$, f is twice-continuously differentiable, $f(\alpha) = 0$, $f'(\alpha) = 0$, and $f''(\alpha) > 0$. Specifically, consider the algorithm

$$x_{k+1} = \begin{cases} x_k - \frac{f(x_k)}{f'(x_k)} & \text{if } f'(x_k) \neq 0 \\ x_k & \text{otherwise.} \end{cases}$$

for $k = 0, 1, \ldots$ with a starting point x_0 sufficiently close to α. Let $e_k = x_k - \alpha$ for $k = 0, 1, 2, \ldots$. Show linear rates convergence through the following steps.

(a) Show that

$$|e_{k+1}| \leq \frac{1}{2}|e_k| + o(e_k)$$

for small e_k.

(b) Assume $e_k \to 0$ as $k \to \infty$. Show that

$$|e_k| \leq \mathcal{O}\Big(\big(\tfrac{1}{2} + \varepsilon\big)^k\Big)$$

for any $\varepsilon > 0$.

(c) Further assume f is thrice-continuously differentiable. Show that

$$|e_{k+1}| \leq \frac{1}{2}|e_k| + O(e_k^2)$$

for small e_k.

(d) Further assume f is thrice-continuously differentiable. Assume $e_k \to 0$ as $k \to \infty$. Show that

$$|e_k| \leq \mathcal{O}\Big(\frac{1}{2^k}\Big),$$

i.e. show that $\limsup_{k\to\infty} |e_k| 2^k < \infty$.

Hint. For part (a), start with the following applications of Taylor's theorem:

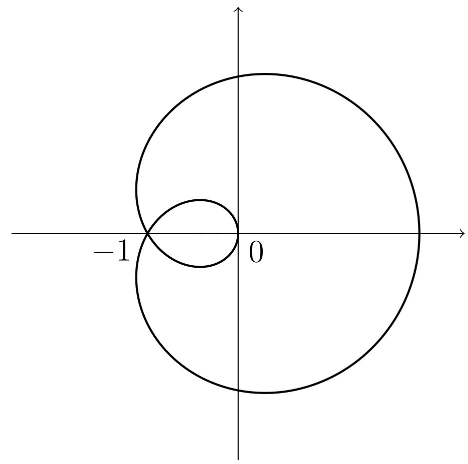
$$f(x_k) = \frac{1}{2}f''(\alpha)e_k^2 + o(e_k^2) \qquad \text{and} \qquad f'(x_k) = f''(\alpha)e_k + o(e_k).$$

Problem [4] (10 Pts.) Draw the region of absolute stability for the 2-step linear multistep method

$$y_{n+2} - y_{n+1} = hf_n.$$

Hint. Consider the following curve

$$\{(\cos(2\theta) - \cos(\theta), \sin(2\theta) - \sin(\theta)) \in \mathbb{R}^2 \mid \theta \in [0, 2\pi)\} =$$



Problem [5] (10 Pts.) Consider the system of two equations for $U(x, y, t) = \begin{bmatrix} u_1(x, y, t) \\ u_2(x, y, t) \end{bmatrix}$,

$$\frac{\partial U}{\partial t} = A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y},$$

to be solved for $t > 0$ in the box $0 < x < 1$, $0 < y < 1$. Smooth initial data, $U(x, y, 0) = g(x, y)$, is given. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) What boundary conditions of the form $au_1 + bu_2 = 0$ can be given at the four boundary planes, $x = 0$, $x = 1$, $y = 0$, $y = 1$, with four different pairs of constants (a, b) will give well-posedness of this initial-boundary value problem?

(b) Construct a convergent finite difference approximation to this problem.

Justify your answers.

Problem [6] (10 Pts.) Consider the convection-diffusion equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = b\frac{\partial^2 u}{\partial x^2}$$

to be solved for $t > 0$, a, b, constants, $b > 0$, $0 \le x \le 1$, with periodic boundary conditions and smooth initial data $u(x, 0) = g(x)$.

Construct a convergent finite difference scheme whose time step is of the order of the space increment.

Justify your answer.

Problem [7] (10 pts) Consider the boundary-value problem

$$\begin{aligned} -\triangle u = f(x,y) \quad & \text{for} \quad (x,y) \in \Omega, \\ u = 10 \quad & \text{for} \quad (x,y) \in \partial\Omega_1 \\ \frac{\partial u}{\partial \vec{n}} + u = y \quad & \text{for} \quad (x,y) \in \partial\Omega_2, \end{aligned}$$

where

$$\begin{aligned} \Omega &= \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 < 1\}, \\ \partial\Omega_1 &= \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1,\ y > 0\}, \\ \partial\Omega_2 &= \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1,\ y \le 0\}, \end{aligned}$$

$f \in L^2(\Omega)$, and $\vec{n}$ is the exterior unit normal to $\partial\Omega$.

(a) Derive a weak variational formulation of the problem by choosing the appropriate space of test functions V.

(b) Obtain existence and uniqueness of solutions to the weak variational formulation by verifying the assumptions of the Lax-Milgram Theorem.

(c) Develop and describe a piecewise-linear Galerkin finite element approximation of the problem and a set of basis functions that spans V_h, such that the corresponding linear algebraic system is sparse. Show that this linear system has a unique solution u_h.

(d) Prove a qualitative error estimate of the form

$$\|u - u_h\|_V \le \|u - v\|_V, \text{ for all } v \in V_h.$$

(e) State a (quantitative) rate of convergence of your approximation.

Justify your answers.