

Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: _____

DATE: _____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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Pass/fail recommend on this form.

Total score: _____

Spring 2026, ADE Qual

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Find two independent series solutions of the ordinary differential equation

$$4t \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 3y = 0,$$

and also prove explicitly using a convergence test the values of $t \in \mathbb{R}$ for which each series converges. Briefly comment on the differentiability and analyticity properties of the series.

2. Let $\Omega = [-\pi, \pi]^2$ and let T^2 denote the corresponding flat 2D torus obtained by imposing periodic boundary conditions on Ω . Consider the Poisson equation:

$$-\Delta u = f \quad \text{in } T^2 \tag{1}$$

- (a) First consider the case where $f(x) = 0$. Show that the only solutions of the equation that satisfy the periodic boundary conditions have constant u .
- (b) Suppose f is a given smooth function. Prove that a necessary condition for the existence of a classical solution $u \in C^2(T^2)$ is that $\int_{T^2} f \, dx = 0$.
- (c) To construct solutions for arbitrary f , we need a periodic Green's function $G(x)$ satisfying

$$-\Delta G = \delta(x) - A. \tag{2}$$

- (i) With reference to your answer from (b), derive the constant A .
- (ii) Represent $G(x)$ as a double Fourier series $G(x) = \sum_{k \in E} \hat{G}_k e^{ik \cdot x}$, calculate the coefficients \hat{G}_k and the set of admissible wavenumbers, E .
- (iii) Assuming $x \neq 0$, show that the sum in (ii) is not absolutely convergent.
- (iv) Again assuming that $x \neq 0$, show that the sum is conditionally convergent. [**Hint:** you may make use the Abel's inequality in the form that if (a_n) , (b_n) are sequences such that a_n is a decreasing sequence of non-negative real numbers and b_n is real or complex, and has bounded partial sums: $|\sum_{n=1}^N b_n| < M$, then $|\sum_n a_n b_n| < M \max(a_n)$]

3. The displacement of a cantilevered elastic beam, $u(x)$, due to a force density $F(x)$ applied along the length of the beam is described by the Euler-Bernoulli equation:

$$u_{xxxx} = F(x) \quad ; \quad 0 < x < 1 \quad ; \quad u(0) = u_x(0) = u_{xx}(1) = u_{xxx}(1) = 0. \tag{3}$$

- (a) Show that the displacement of the beam minimizes the energy functional

$$E[u] = \frac{1}{2} \int_0^1 (u_{xx})^2 \, dx - \int_0^1 F u \, dx \tag{4}$$

among every C^4 functions whose boundary conditions you should specify.

- (b) Suppose that u_i , $i = 1, 2$ are solutions of the Euler-Bernoulli equation with force densities $F_i(x)$. Show that the two displacements obey the Betti reciprocity principle:

$$\int_0^1 (F_1 u_2 - F_2 u_1) dx = 0 \quad (5)$$

- (c) Hence show that the displacement of the beam also obeys Maxwell reciprocity, namely, if a point force is applied at a point A , then the displacement at a point B is the same as if the point force were applied at B and the displacement measured at A .

4. Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx} \quad , \quad 0 < t < T \quad , \quad x \in I \quad (6)$$

- (a) First consider the case where $I = \mathbb{R}$. Suppose that $u_t(x, 0) = 0$ and $\phi(x) \equiv u(x, 0)$ is compactly supported. Show that $u(x, t)$ obeys a maximum principle, that is: $u(x, t) \leq \max \phi(x)$.
- (b) Now let $I = \{x : x > 0\}$, and impose a boundary condition $u(0, t) = h(t)$, a continuous function. Does $u(x, t)$ still obey a maximum principle? That is, do we have: $\max u(x, t) \leq \max(\max \phi(x), \max h(t))$?

5. $u(x, t)$ solves the wave equation with additional spring force:

$$u_{tt} = \Delta u - u$$

Suppose that $u(x, 0)$, $u_t(x, 0)$ are compactly supported on the ball $B(0, 1)$. Show that the compact support of $u(x, t)$ is contained in $B(0, 1 + t)$.

6. Consider the Ginzburg-Landau energy

$$E(u) = \int \frac{1}{2} |\nabla u|^2 dx + \int W(u) dx$$

for $W = (1 - u^2)^2$ a double-well potential.

- (a) Show that the Allen-Cahn equation

$$u_t = \Delta u - W'(u)$$

is the L^2 gradient descent of the GL energy.

- (b) Show that the Cahn-Hilliard equation

$$u_t = -\Delta(\Delta u - W'(u))$$

is the gradient descent of the GL functional in the H^{-1} inner product where you can assume

$$\nabla_{H^{-1}} E(u) = -\Delta \nabla_{L^2} E(u).$$

(c) prove that the Cahn-Hilliard equation decreases the GL energy while preserving $\int u$.

(d) Show by example that the Allen-Cahn evolution does not preserve $\int u$.

7. Consider solutions of the porous media equation $u_t = (uu_x)_x$ on the torus \mathbf{T}^1 with positive initial data $u_0 > M > 0$. Prove that any smooth solution of this equation for $t > 0$ also satisfies $u(x, t) > M$ for all $x \in T^1$ and $t > 0$.

8. Consider the scalar conservation law

$$u_t + (u^4)_x = 0$$

with initial condition $u = 1$; $0 < x < 1$, and $u = 0$ otherwise.

Solve the initial value problem for all $t > 0$.