Algebra

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________ 5. ______________________

2. ______________________ 6. ______________________

3. ______________________ 7. ______________________

4. ______________________ 8. ______________________

Pass/fail recommend on this form.

Total score: ____________________

Algebra Qualifying Exam
Spring 2026
Complete 8 of the following 10 problems.
(If attempting more than 8, indicate which 8 you wish to have graded.)

1. Let G be a finite group such that for every positive integer m there is at most one subgroup of G of order m. Prove that G is cyclic.

2. Prove that the field extension $\mathbb{Q}(\cos(\frac{\pi}{2^n}))/\mathbb{Q}$ is finite Galois and determine its Galois group.

3. Let A be a free abelian group of countable rank and let $R = \mathrm{End}(A)$. Prove that the left free R-modules R and $R \oplus R$ are isomorphic.

4. Let $F : \mathit{Groups} \to \mathit{Sets}$ be the functor that assigns to a group the set of all its subgroups. Is the functor F representable?

5. Let $0 \to A \to B \to C \to 0$ be an exact sequence and let $f : C \to C'$ be an injective homomorphism of abelian groups. Prove that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

with the bottom row an exact sequence of abelian groups.
Hint: If you do not know Ext, use an embedding of B into an injective abelian group I and consider I/A.

6. Decompose, with proof, the rational group ring $\mathbb{Q}[A_4]$ as a direct sum of simple algebras. (Here, as usual, A_4 is the alternating group on 4 letters, and $\mathbb{Q}[A_4] = \mathbb{Z}[A_4]\otimes_{\mathbb{Z}}\mathbb{Q}$.) You may use facts you know about the subgroups and quotients of A_4 provided you state clearly what you are using. You may also use general facts about simple algebras and group rings which you know.

7. Let a and b be positive integers and let p be a prime number. Let $q_1 = p^a$ and $q_2 = p^b$. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be finite fields having $q_1$and q_2 elements, respectively. Determine, including the degrees and multiplicities of any fields occurring, the structure of the algebra $A = \mathbb{F}_1 \otimes_{\mathbb{Z}} \mathbb{F}_2$. You may quote any general results you know about tensor products of fields.

8. Classify, up to isomorphism, all 3 dimensional modules over the polynomial ring $\mathbb{F}_2[X]$ which are not semisimple. Here $\mathbb{F}_2$ is the field with 2 elements. How many modules are there in total?

9. Let V_n be the standard n-dimensional representation of the symmetric group S_n (i.e. the one gotten by the permuting the entries of vectors in $\mathbb{C}^n$). Decompose V as a direct sum of irreducibles, proving that each summand is irreducible.

10. Let R be a commutative ring. Prove that an element r of R which belongs to all prime ideals of R is nilpotent.