Analysis

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

****

1. ______________________
2. ______________________
3. ______________________
4. ______________________
5. ______________________
6. ______________________
7. ______________________
8. ______________________

Pass/fail recommend on this form.

Total score: ____________________

ANALYSIS QUALIFYING EXAM: MARCH 26, 2026

In order to pass the exam, you need to show mastery of both real and complex analysis. Choose ten problems to work on, including five from Problems 1–6 and five from Problems 7–12. Please indicate on the front of your paper which ten problems you wish to have graded.

Problem 1: Let X be a complete separable metric space and let μ be a positive and finite Borel measure on X. Prove that for each $\epsilon > 0$, there exists a compact set $K \subseteq X$ such that $\mu(X \smallsetminus K) < \epsilon$.

Problem 2: Let $a < b$ be real numbers and, for each natural $n \geq 1$, let $F_n \colon [a,b] \to \mathbb{R}$ be a non-decreasing continuously differentiable function. Assume that the infinite series $G(x) := \sum_{n=1}^{\infty} F_n(x)$ converges for all $x \in [a,b]$. Prove that G is continuous on $[a,b]$ and

$$G'(x) \text{ exists} \quad \text{and} \quad G'(x) = \sum_{n \geq 1} F_n'(x)$$

holds for Lebesgue a.e. $x \in (a,b)$.

Problem 3: Let $f \in L^1(\mathbb{R}^d)$ be non-negative such that $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ has positive Lebesgue measure. Prove that for any sequence of points $\{x_n\}_{n \in \mathbb{N}}$ that is dense in $\mathbb{R}^d$,

$$\sum_{n=1}^{\infty} f(x + x_n) = \infty$$

holds for Lebesgue a.e. $x \in \mathbb{R}^d$.

Problem 4: Let $L^2(\mathbb{R})$ be defined with respect to the Lebesgue measure on $\mathbb{R}$. Prove that there exists a continuous linear operator $T \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that for each continuously differentiable function $f \colon \mathbb{R} \to \mathbb{R}$ with compact support,

$$Tf(x) = \int_0^{\infty} \frac{f(x+t) - f(x-t)}{t} \, \mathrm{d}t$$

holds for a.e. $x \in \mathbb{R}$.

Problem 5: Write $\langle f, g \rangle := \int fg \mathrm{d}\lambda$, where λ is the Lebesgue measure on $[0,1]$, whenever the integral exists. Let $L^1([0,1])$ be the space of real-valued measurable functions $f \colon [0,1] \to \mathbb{R}$ such that $\|f\|_1 := \int |f| \mathrm{d}\lambda < \infty$. For each natural $n \geq 1$, denote $e_n(x) := \mathrm{e}^{2\pi i n x}$. Prove the following:

(a) For each $f \in L^1([0,1])$,

$$\lim_{n\to\infty} \langle f, e_n\rangle = 0.$$

(b) For each $A > 0$ there exists $f \in L^1([0,1])$ such that

$$\|f\|_1 > A \sup\bigl\{|\langle f, e_n\rangle| \colon n \ge 1\bigr\}.$$

Hint: Consider, e.g., f equal to $x^{-\alpha}$ for $x \le 1/2$ and equal to $-(1-x)^{-\alpha}$ for $x > 1/2$, where α varies through $(0,1)$.

(c) There exists a complex-valued sequence $\{a_n\}_{n\ge1}$ with $\lim_{n\to\infty} a_n = 0$ which is different from $\{\langle f, e_n\rangle\}_{n\ge1}$ for all $f \in L^1([0,1])$.

Problem 6: Let X be a non-empty set and let $\mathcal{X}$ be the vector space of bounded real-valued functions on X endowed with the supremum norm $\|f\| := \sup_{x\in X} |f(x)|$. Let $\phi\colon \mathcal{X} \to \mathbb{R}$ be a continuous linear functional. For $f \in \mathcal{X}$ with $f \ge 0$ set

$$\phi'(f) := \sup\bigl\{\phi(u) \colon u \in \mathcal{X},\, 0 \le u \le f\bigr\},$$

and let $\phi^+ \colon \mathcal{X} \to \mathbb{R}$ be defined by

$$\phi^+(f) := \phi'\bigl(f + \|f\|1\bigr) - \|f\|\phi'(1),$$

where "1" is the function equal to one everywhere. Prove the following:

(a) ϕ^+ and $\phi^- := \phi^+ - \phi$ are continuous linear functionals on $\mathcal{X}$.

(b) $\phi^\pm$ are positive in the sense that $f \ge 0$ implies $\phi^\pm(f) \ge 0$, and minimal in the sense that for any positive continuous linear functionals $\psi, \chi\colon \mathcal{X} \to \mathbb{R}$ such that $\phi = \psi - \chi$, we have

$$\phi^+(f) \le \psi(f) \quad \text{and} \quad \phi^-(f) \le \chi(f)$$

for all $f \ge 0$.

Problem 7: Let $f \colon \{z \in \mathbb{C} \colon |z| < 2\} \to \mathbb{C}$ be a holomorphic function satisfying $f(0) = 0$, $f(1) = 1$, and $|f(z)| \le 1$ for all $|z| \le 1$. Show that:

(a) $f'(1) \in \mathbb{R}$. Hint: Consider the function $\theta \mapsto |f(\mathrm{e}^{\mathrm{i}\theta})|^2$.

(b) $f'(1) \ge 1$.

Problem 8: Let $\Omega \subseteq \mathbb{C}$ be open and let $f_j : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions such that, for some constant $M \ge 0$ and all $z \in \Omega$,

$$F(z) := \sum_{j=1}^{\infty} |f_j(z)|^2 \le M.$$

Prove the following:

(a) the series $\sum_{j=1}^{\infty} |f_j(z)|^2$ converges uniformly on compact subsets of Ω.

(b) the function F is subharmonic in Ω.

Problem 9: Let $\mathcal{D}$ be the space of functions $f(z)$ holomorphic in $z = x + iy \in \mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}x\mathrm{d}y < \infty.$$

The space $\mathcal{D}$ is a Hilbert space when equipped with the inner product

$$(f, g) = f(0)\overline{g(0)} + \frac{1}{\pi} \int_{\mathbb{D}} f'(z)\,\overline{g'(z)}\,\mathrm{d}x\mathrm{d}y.$$

Show that for each $a \in \mathbb{D}$, the map $\mathcal{D} \ni f \mapsto f(a) \in \mathbb{C}$ is a linear continuous functional on $\mathcal{D}$ and find an explicit function $K_a \in \mathcal{D}$ such that $f(a) = (f, K_a)$ for all $f \in \mathcal{D}$.

Problem 10: Show that

$$\frac{1}{\mathrm{e}^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4n^2\pi^2}$$

for all $z \in \mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$, with the series in the right hand side converging uniformly on compact subsets of $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$.

Problem 11: Let $u : \mathbb{D} \to \mathbb{R}$ be a continuous subharmonic function on the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ such that $u < 0$ on $\mathbb{D}$. Show that for each $\zeta \in \partial\mathbb{D}$,

$$\limsup_{r \to 1^-} \frac{u(r\zeta)}{1 - r} < 0.$$

Problem 12: Let $h \colon \mathbb{C} \to \mathbb{C}$ be defined by

$$h(z) := \int_0^z \mathrm{e}^{-w^2} \,\mathrm{d}w,$$

where the integration is along the linear segment from 0 to z.

(a) Show that h is an entire function such that $h(\mathbb{C}) = \mathbb{C}$. Hint: Show first that h is an odd function.

(b) Show that for each $b \in \mathbb{C}$ and each $a \in h^{-1}(b)$ there exists a neighborhood $U \subseteq \mathbb{C}$ of b and a holomorphic function $g : U \to \mathbb{C}$ such that $g(b) = a$ and $h(g(z)) = z$ for all $z \in U$.

(c) Show that there is no holomorphic function g on $\mathbb{C}$ such that $h(g(z)) = z$ for all $z \in \mathbb{C}$.
