BASIC EXAM: SPRING 2026

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam!!!**

All answers must be justified. If you wish to use a known theorem, make sure to give a full and precise statement.

Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want us to grade.

Important: No books, notes, calculators, computers of other printed or electronic materials can be used on the exam.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1
Let $A \in M_n(\mathbb{C})$ be a diagonalizable matrix. Let $m_1, \dots, m_r$ be the dimensions of its distinct eigenspaces. Let W be a subset of matrices in $M_n(\mathbb{C})$ that commute with A. Prove that W is a subspace of $M_n(\mathbb{C})$ of dimension $m_1^2 + \cdots + m_r^2$.

Problem 2
Let V be a finite-dimensional vector space over a field F, and let $T\colon V \to V$ be a linear transformation. Prove that there exists a positive integer k such that

$$V = \ker(T^k) \oplus \operatorname{im}(T^k).$$

Problem 3
Let A be an $n \times n$ real skew-symmetric matrix. Prove that every nonzero eigenvalue of A is purely imaginary and that $\det(A + I_n) \geq 1$, where I_n is an identity $n \times n$ matrix.

Problem 4
Let $A \in M_4(\mathbb{R})$ be a real 4×4 matrix which satisfies $A^T = -A$. Prove that there exists an orthonormal basis of $\mathbb{R}^4$ in which the matrix of A has the form

$$\begin{pmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{pmatrix}$$

for some $a, b \geq 0$.

Problem 5
Let V be a finite-dimensional real vector space. Let $a\colon V \times V \to \mathbb{R}$ be a symmetric and positive semidefinite bilinear form, and let $L\colon V \to \mathbb{R}$ be a linear function. Set

$$J(v) = \frac{1}{2}a(v, v) - L(v)$$

for all $v \in V$. Show that $u \in V$ is a solution to $a(u, v) = L(v)$ for all $v \in V$ if and only if it is a solution to $J(u) = \inf_{v \in V} J(v)$.

Problem 6
Let $A \in M_n(\mathbb{C})$, $n \geq 2$, be in Jordan canonical form, and suppose its minimal polynomial is $(x - \lambda)^n$ for some $\lambda \in \mathbb{C}$. Let B be obtained from A by replacing its $(n, 1)$-entry by 1. Find the Jordan canonical form of B.

Problem 7
Let $f \in C[a,b]$ be such that $\operatorname{im}(f) \subset [a,b]$. Suppose, in addition, that f' is continuous on (a,b) and there exists a positive constant $k < 1$ such that $|f'(x)| \leq k$ for all $x \in (a,b)$. Furthermore, let $x_0 \in [a,b]$, and define $x_n = f(x_{n-1})$ for each $n \geq 1$.

(1) Prove that there exists a unique $p \in [a,b]$ such that $f(p) = p$, and the sequence $\{x_n\}_{n=0}^{\infty}$ converges to p.

(2) Show that if $f'(p) \neq 0$, then

$$\lim_{n\to\infty} \left| \frac{x_n - p}{x_{n-1} - p} \right| > 0$$

Problem 8
Let B be a closed ball of positive radius in $\mathbb{R}^n$, with $n > 1$. Prove that a continuous function from B to $\mathbb{R}$ cannot be injective.

Problem 9
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C^{\infty}([0,1])$, with following property: for each $m \geq 0$, there exists some B_m such that

$$\text{for all } n \text{ and all } x \in [0,1], \quad |f_n^{(m)}(x)| \leq B_m$$

(Here $f_n^{(m)}$ refers to the m^{th} derivative of f_n.)
Prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ and all of its derivatives converge uniformly on $[0,1]$.

Problem 10
Let (X,d) be a metric space, and let $E \subset X$ be compact. Suppose that for all $\epsilon > 0$ and every $a, b \in E$, there exist a positive integer n and a set of points $a = x_0, x_1, \ldots, x_n = b \in E$ such that $d(x_{k-1}, x_k) < \epsilon$ for all $1 \leq k \leq n$.

(1) Prove that E is connected.

(2) Give an example to show that E may not be path-connected.

Problem 11
Let (X,d) be a compact metric space, and for each $n \in \mathbb{N}$, let $f_n \colon X \to \mathbb{R}$ be a continuous function. Furthermore, assume that for each $x \in X$, the sequence $\{f_n(x)\}_{n=0}^{\infty}$ of real numbers is non-increasing and converges to 0. For each $N \in \mathbb{N}$, define $s_N \colon X \to \mathbb{R}$ by

$$s_N(x) = \sum_{n=0}^{N} (-1)^n f_n(x)$$

Prove that for each $x \in X$, the sequence $\{s_N(x)\}$ converges, and that the function $s(x) = \lim_{N\to\infty} s_N(x)$ is a continuous function on X.

Problem 12

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable, and let $C = f(x_0, y_0, z_0)$.

(1) Give a sufficient condition under which the level set $f(x, y, z) = C$ can be expressed as *all* of the following:

$$\begin{aligned} z &= z(x, y) \quad \text{for } (x, y) \text{ in some neighborhood of } (x_0, y_0),\\ y &= y(x, z) \quad \text{for } (x, z) \text{ in some neighborhood of } (x_0, z_0), \text{ and}\\ x &= x(y, z) \quad \text{for } (y, z) \text{ in some neighborhood of } (y_0, z_0), \end{aligned}$$

with each of the functions x, y, z being C^1 functions. Justify your answer.

(2) Prove that $\left.\dfrac{\partial x}{\partial y}\right|_{(y_0,z_0)} \left.\dfrac{\partial y}{\partial z}\right|_{(x_0,z_0)} \left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)} = -1.$