Geometry/Topology

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ______________________________

DATE: ______________________________

EXAMINEES: DO NOT WRITE BELOW THIS LINE

****

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________

Pass/fail recommend on this form.

Total score: ____________________

QUALIFYING EXAM

Geometry/Topology

Spring 2026

Answer all 10 questions. Each problem is worth 10 points. You need to provide sufficient justification for each problem. In order to pass, you must receive 60/100 AND (get 8/10 on 6 problems OR (get 8/10 on 5 problems AND get 5/10 on 2 other problems)).

(Q-1) Derive Thom's Transversality Theorem from Sard's Theorem. That is, let X, P, M be smooth manifolds and $N \subset M$ be a smooth embedded submanifold. Let $f\colon X \times P \to M$ be a smooth map, transverse to N. Use Sard's Theorem to prove that the set

$$\{t \in P \mid f|_{X\times t} \text{ is not transverse to } N\}$$

is a measure zero subset of P.

(Q-2) You may not use Whitney embedding theorem for this problem, but you may use the existence of partition of unity. Let M be a compact smooth n-dimensional manifold.

- (a) (4 pts) Prove that there is an immersion $\iota\colon M \to \mathbb{R}^k$ for some k.
- (b) (6 pts) If $k > 2n$, prove that there is a $(k-1)$-dimensional subspace $H \subset \mathbb{R}^k$ such that $\pi \circ \iota$ is an immersion, where $\pi\colon \mathbb{R}^k \to H$ is the orthogonal projection.

(Q-3) Let $f_i\colon M \to N$, $i = 0, 1$, be two smooth maps between smooth manifolds M and N, and $f_i^*\colon \Omega^*(N) \to \Omega^*(M)$, $i = 0, 1$, be the induced chain maps between the respective de Rham complexes.

- (a) (2 pts) Define the notion of a smooth homotopy between f_0 and f_1.
- (b) (2 pts) Define the notion of a chain homotopy between f_0^* and f_1^*.
- (c) (6 pts) Prove that a smooth homotopy from f_0 to f_1 induces a chain homotopy between f_0^* and f_1^*.

(Q-4) Let M be a smooth manifold and ω a nowhere vanishing 1-form on M. Prove the following are equivalent.

- (a) $\omega \wedge d\omega = 0$.
- (b) Around every point $p \in M$ there is a neighborhood U and functions $f, \lambda\colon U \to \mathbb{R}$ such that $\omega = \lambda df$ on U.
- (c) Around every point $p \in M$ there is a neighborhood U and a 1-form η on U such that $d\omega = \eta \wedge \omega$ on U.

(Q-5) Consider the form

$$\omega = (x^{2026} + x + y + z)dy \wedge dz$$

on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $\iota\colon S^2 \to \mathbb{R}^3$ the inclusion map. Construct a closed 2-form α on $\mathbb{R}^3$ such that $\iota^*\alpha = \iota^*\omega$, or show that such a form α does not exist.

(Q-6) Find all topological spaces X such that $\mathbb{C}P^{2026}$ is a covering space for X.

(Q-7) Let $\mathrm{SO}(n) = \{n \times n \text{ matrix } A \mid A^T A = I, \det(A) = 1\} \subset \mathbb{R}^{n^2}$.

- (a) (3 pts) Prove that $\mathrm{SO}(n)$ is a compact smooth manifold and compute its dimension N.
- (b) (2 pts) Compute $H_N(\mathrm{SO}(n);\mathbb{Z})$, where N is the dimension of $\mathrm{SO}(n)$.
- (c) (3 pts) Let $n \geq 3$. Prove $\pi_1(\mathrm{SO}(n), I) \cong \mathbb{Z}/2$. (Hint: You may use induction and the following fact: If $E \to B$ is a fiber bundle with fiber F, and B is a CW complex with 2-skeleton a point, then the inclusion $F \to E$ induces isomorphism on π_1.)
- (d) (2 pts) For $n \geq 3$, is the universal cover of $\mathrm{SO}(n)$ same as its orientation double cover?

(Q-8) Let $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-20)(x-26)\}$.

- (a) (5 pts) Compute $H_*(X;\mathbb{Z})$.
- (b) (5 pts) Prove X is path-connected and compute $\pi_1(X, x)$ for any $x \in X$.

(Hint: Consider the projection $X \to \mathbb{C}$ to the x-coordinate.)

(Q-9) Let X be a connected 2-dimensional CW complex with a single 2-cell. Assume its universal cover is contractible. Prove that any connected subcomplex of X also has the property that its universal cover is contractible.

(Q-10) Let $X = \{A_0, A_1, B_0, B_1, C_0, C_1\}$ be a partially ordered set:

$$A_i < B_j < C_k \qquad \text{for all } i, j, k \in \{0, 1\}.$$

Let $U_x = \{y \in X \mid x \leq y\}$ and arbitrary unions of these U_x be open sets.

- (a) (3 pts) Check that the open sets define a topology on X.
- (b) (3 pts) Prove that U_{A_0} is contractible.
- (c) (4 pts) Compute $H_*(X;\mathbb{Z})$.