Logic

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________ 5. __________________________

2. ______________________ 6. __________________________

3. ______________________ 7. __________________________

4. ______________________ 8. __________________________

Pass/fail recommend on this form.

Total score: ___________________

Qualifying Examination
LOGIC
Spring 2026

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each question, you may use previous questions if needed even if you did not answer them.

We use $\ulcorner\varphi\urcorner$ for the Gödel number of φ.

Problem 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and consider the structure $\mathcal{R} = (\mathbb{R}, <, f)$. Suppose $\mathcal{R}^* = (\mathbb{R}^*, <^*, f^*)$ is an $\aleph_1$-saturated elementary extension of $\mathcal{R}$. Call an element $x \in \mathbb{R}^*$ *infinitesimal* if $-a <^* x <^* a$ for all positive $a \in \mathbb{R}$. Show that the function f is continuous at 0 if and only if for all infinitesimal elements $x \in \mathbb{R}^*$, $f^*(x)$ is also infinitesimal.

Problem 2. Let $\mathcal{A}$ be an infinite L-structure in a countable language L and suppose that $\mathrm{Th}(\mathcal{A})$ is $\aleph_0$-categorical. Show that every substructure of $\mathcal{A}$ generated by a finite set is finite.

Problem 3. Does there exist a recursive set $A \subset \omega$ such that for every sentence φ in the language of arithmetic, if $\mathsf{PA} \vdash \varphi$, then $\ulcorner\varphi\urcorner \in A$, and if $\mathsf{PA} \vdash \neg\varphi$, then $\ulcorner\varphi\urcorner \notin A$? (There are no constraints if φ is independent from PA.)

Problem 4. Let T be an r.e. theory in a finite language that is κ-categorical for some $\kappa \geqslant \aleph_0$.

(4a) Show that if T has no finite models, then T is decidable.

(4b) Give an example of such a theory that has finite models and is not decidable.

(For **(4b)**, you may use a language of your choice.)

Problem 5. Let $f\colon \omega \dashrightarrow \omega$ be a partial recursive function defined on an infinite set. Show that there exists a nondecreasing total recursive function $g\colon \omega \to \omega$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$.

Problem 6. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function with the following property: for every continuous function $g\colon \mathbb{R} \to \mathbb{R}$, there is a point $y \in \mathbb{R}$ such that $f(x, y) = g(x)$ for all $x \in \mathbb{R}$. Show that f cannot be continuous.

Problem 7. Let $S \subseteq \omega_1$ be stationary. Prove that there are uncountably many pairwise disjoint stationary subsets of S.

Problem 8. Prove that for each ordinal α, there is at most one ordinal $\beta > \alpha$ such that:

(1) L_β satisfies $\mathsf{ZFC} - \mathsf{Powerset}$,

(2) $L_\beta \models$ "α is regular uncountable", and

(3) for some finite set $p \subseteq L_\beta$, the only $X \preccurlyeq L_\beta$ such that $p \subseteq X$ and $X \cap \alpha$ is an ordinal, is L_β itself.

In fact, prove the stronger statement that there are no ordinals $\gamma > \beta > \alpha$ such that γ satisfies (1) and (2) and β satisfies (3) for the same α.