Student ID Number: ______________________________

UCLA MATHEMATICS - NUMERICAL ANALYSIS: SPRING 2026

INSTRUCTIONS and GRADING:
There are 7 problems. Problems 1-2 are worth 5 points and problems 3-7 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems $[1-4]$ and $[5-7]$ to obtain a passing score.

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. You have 4 hours. Good luck.

NOTATION:

#	Score	Points Possible
1	______	5
2	______	5
3	______	10
4	______	10
5	______	10
6	______	10
7	______	10
Total	______	60

1. (5 Pts.) Consider a quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2),$$

where $w_1, w_2 \in \mathbb{R}$ are weights and $x_1, x_2 \in [-1, 1]$ are nodes.

(a) Determine the explicit values of w_1, w_2, x_1, and x_2 such that the formula is exact for all polynomials $f(x)$ of degree less than or equal to 3.

(b) State the degree of precision of the quadrature rule you obtained and justify your answer.

2. (5 Pts.) Let $f \in C^3(I)$ for some open interval $I \subset \mathbb{R}$ containing x_0, and let

$$x_1 = x_0 + h, \qquad x_2 = x_0 + 2h,$$

where $h > 0$ is sufficiently small so that $x_0, x_1, x_2 \in I$. We seek constants a, b, c (depending on h) such that

$$af(x_0) + bf(x_1) + cf(x_2)$$

approximates $f'(x_0)$ with error $O(h^2)$ as $h \to 0$.

(a) Determine a, b, c in terms of h. Write the resulting finite difference formula explicitly.

(b) Determine the term of order h^2 in the truncation error. Write this term explicitly, including its constant coefficient and its dependence on f, and do not use big-O notation.

3. (10 Pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be analytic near a simple root r (i.e., $f(r) = 0$, $f'(r) \neq 0$). Define the Newton iteration

$$x_{n+1} = \Phi(x_n),$$

where

$$\Phi(x) = x - \frac{f(x)}{f'(x)}.$$

(a) Show that $\Phi'(r) = 0$.

(b) Let $e_n := x_n - r$. By expanding f and f' in Taylor series about r, derive the asymptotic error relation

$$e_{n+1} = \frac{f''(r)}{2f'(r)} e_n^2 + O(e_n^3).$$

(c) Now suppose f has a root of multiplicity $m \geq 2$ at r such that

$$f(x) = (x - r)^m g(x), \qquad g(r) \neq 0.$$

Show that

$$\Phi'(r) = \frac{m-1}{m},$$

and explain why this implies Newton's method converges only linearly asymptotically.

4. (10 Pts.) Consider the ODE

$$\frac{dy}{dt} = f(t, y),$$

and the numerical method

$$\frac{y_{i+1} - y_i}{h} = \xi f(t_i, y_i) + (1 - \xi) f(t_{i+1}, y_{i+1}),$$

where $\xi \in [0, 1]$ is a fixed number. Here $t_0 = 0$, and y_i is an approximation of $y(t_i)$. We use a uniform grid so that $t_{i+1} - t_i = h$.

(a) When is the method explicit and when is it implicit? What is the order of accuracy of this method?
(b) Suppose $f(t, y) = \lambda y$. Discuss the absolute stability property of the method for the case $\lambda < 0$ and $\xi = 0, \frac{1}{2}, 1$.
(c) When the above method is implicit, solving for y_{i+1} may be computationally expensive. We can use a predictor-corrector method by replacing y_{i+1} in $f(t_{i+1}, y_{i+1})$ with $y_i + hf(t_i, y_i)$. Analyze the stability properties of the resulting predictor-corrector method for $\xi = \frac{1}{2}$.

5. (10 Pts.) Consider the scalar partial differential equation

$$u_t + f(u)_x = 0,$$

to be solved for $t > 0$ and u periodic in x with $-1 < x < 1$, and

$$f(u) = \frac{1}{2}u^2 + u.$$

Write a convergent finite difference approximation for the initial value problems:

(a)

$$u(x, 0) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

(b)

$$u(x, 0) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases}$$

Justify your answers. Describe the qualitative structure of the solutions.

6. (10 Pts.) Consider the equation

$$u_t + \sin x \, u_x - y \, u_y = 0,$$

to be solved for $t > 0$, $(x, y) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $u(x, y, 0) = \psi(x, y)$ given.

(a) On what part of the square boundary should u be specified? Why?
(b) Construct a stable and convergent scheme approximating this problem. Justify your answers.

7. (10 pts.) Consider the boundary value problem

$$
\begin{aligned}
-\Delta u + u &= f(x,y), && (x,y) \in \Omega,\\
u &= 0, && (x,y) \in \partial\Omega_1,\\
\frac{\partial u}{\partial n} + u &= y, && (x,y) \in \partial\Omega_2,
\end{aligned}
$$

where f is a given function on Ω and n denotes the outward unit normal to $\partial\Omega$. Let

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\},$$

be the domain and split the boundary into two parts:

$$\partial\Omega_1 = \{(x,y) \in \partial\Omega \mid x \le 0\}, \qquad \partial\Omega_2 = \{(x,y) \in \partial\Omega \mid x > 0\}.$$

(a) Derive a weak (variational) formulation of the problem. Specify the trial and test spaces, the bilinear form $a(\cdot,\cdot)$, and the linear functional $L(\cdot)$.

(b) State conditions on f under which existence and uniqueness of a weak solution follow from the Lax-Milgram theorem. Clearly state the coercivity and boundedness properties you use.

(c) Describe a Galerkin finite element method using continuous, piecewise linear (P_1) elements on a triangulation of Ω. Explain how the Dirichlet condition on $\partial\Omega_1$ is imposed and how the Robin boundary condition contributes to the discrete system.

(d) What would be a standard error estimate for Part (c) as function of the meshsize h?