Optimization / Numerical Linear Algebra

(ONLA)

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: ____

DATE: ____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ______________________ 5. __________________________

2. ______________________ 6. __________________________

3. ______________________ 7. __________________________

4. ______________________ 8. __________________________

Pass/fail recommend on this form.

Total score: ____________________

Qualifying Exam, Spring 2026

Optimization / Numerical Linear Algebra (ONLA)

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. PLEASE USE BLANK PAGES AT END FOR ADDITIONAL SPACE.

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1. (10 points) Assume the Axiom of Floating Point Arithmetic (on $\mathbb{R}$) holds with machine precision $\varepsilon_{\text{machine}}$. Denote by $\oplus$ and $\ominus$ floating point addition and subtraction, respectively.

   a) Prove that the same Axiom also holds for complex-valued subtraction, with $\varepsilon := C\varepsilon_{\text{machine}}$ for a constant C (explicitly calculate an appropriate C in your proof).

   b) Define the floating point algorithm $\hat{f}(x, y) = \text{fl}(x) \oplus \text{fl}(y)$ for $x, y \in [0, \infty)$. Is this algorithm stable, backward stable, both, or neither? Prove your answers.

   c) Define the floating point algorithm $\hat{f}(x) = \text{fl}(x) \oplus 1$ for $x \in [0, \infty)$. Is this algorithm stable, backward stable, both, or neither? Prove your answers (and don't forget to consider small $x \approx 0$).

2. (10 points) Consider solving the least squares problem $\min_x \|Ax - b\|_2^2$ for full-rank $m \times n$ matrix $(m \geq n)$.

   a) Assume A is a fixed diagonal matrix (with extra zero rows if $m > n$). Consider the problem that takes as input b and outputs the least squares minimizer x. Prove that the relative condition number (with respect to the Euclidean norm) of this problem is $\frac{\kappa(A)}{\eta \cos\theta}$, where $\eta = \frac{\|A\|_2 \|x\|_2}{\|Ax\|_2}$, $\theta = \cos^{-1} \frac{\|Ax\|_2}{\|b\|_2}$, and $\kappa(A) = \|A\|_2 \|A^\dagger\|_2$ is the condition number of A.

   b) Now let A be an arbitrary full rank $m \times n$ matrix $(m \geq n)$. Consider the same problem as in a). Prove that the problem has the same condition number as in part a) in this general case as well.

3. (10 points) Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ have full column rank, and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ denote the singular values of A. Consider the least squares problem

$$\min_x \|Ax - b\|_2,$$

with exact (infinite precision) solution $x^\star$.

a) Write x^* in terms of the SVD of $A = U\Sigma V^*$.

b) Denote again the condition number $\kappa(A) = \sigma_1/\sigma_n$. Show that the condition number of the normal equations matrix satisfies

$$\kappa(A^*A) = \kappa(A)^2.$$

Briefly explain the numerical implication of this result for solving the least squares problem by forming the normal equations and applying a Cholesky factorization.

c) Assume that a computed solution $\widehat{x}$ is the exact solution of a perturbed problem

$$(A + \Delta A)\widehat{x} = b + \delta b,$$

where the perturbations satisfy

$$\frac{\|\Delta A\|_2}{\|A\|_2} = O(\varepsilon), \qquad \frac{\|\delta b\|_2}{\|b\|_2} = O(\varepsilon),$$

with ε denoting machine precision. Show that

$$\frac{\|\widehat{x} - x^\star\|_2}{\|x^\star\|_2} = O\big(\kappa(A)\,\varepsilon\big).$$

Explain informally why solving the least squares problem via the normal equations typically leads to a forward error bound of order $O\big(\kappa(A)^2\varepsilon\big)$.

4. (10 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite, and consider a matrix splitting $A = M - N$, where M is nonsingular. Define the associated iteration matrix $T = M^{-1}N$. Assume that $M + N$ is symmetric positive definite. Write the A–inner product, A–norm (for vector x) and A–norm (for matrix B) by

$$\langle x, y\rangle_A := x^T Ay, \qquad \|x\|_A := \sqrt{x^T Ax}, \qquad \|B\|_A := \sup_{x\neq 0} \frac{\|Bx\|_A}{\|x\|_A}.$$

a) Prove that T is self-adjoint with respect to the A-inner product.

b) Show that the stationary iteration
$$x^{(k+1)} = Tx^{(k)} + M^{-1}b$$
is a contraction in the A-norm if and only if $\rho(T) < 1$. (You may use the fact that it is a contraction if and only if $\|T\|_A < 1$).

c) Explain whether or not this would hold for an arbitrary matrix norm.

5. (10 points) Let $S \in \mathbb{R}^{n \times n}$ be symmetric and satisfy

$$\|S\|_2 \le 0.5.$$

Define

$$A = I + S.$$

(a) Show that A is symmetric positive definite

(b) Consider the linear system

$$Ax = b$$

solved by the Conjugate Gradient method with initial guess $x^{(0)} = 0$. Give an upper bound on the number of CG iterations k required to guarantee

$$\frac{\|x - x^{(k)}\|_A}{\|x\|_A} \le 10^{-15}.$$

6. (10 points) Let an iterative method be defined by

$$x^{(k+1)} = Gx^{(k)} + c,$$

where $G \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}^n$.

Assume $x^\star$ is a fixed point of the iteration, i.e.

$$x^\star = Gx^\star + c.$$

Define the error by

$$e^{(k)} = x^\star - x^{(k)}.$$

(a) Show that the error satisfies

$$e^{(k+1)} = Ge^{(k)}.$$

(b) Show that if $\rho(G) < 1$, then the iteration converges to $x^\star$ for any initial guess.

(c) Show that if there exists a matrix norm such that $\|G\| < 1$, then the iteration converges to $x^\star$.

(d) Give an example of a matrix G such that $\rho(G) < 1$ but $\|G\|_2 > 1$.

7. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and suppose $x^\star \in \mathbb{R}^n$ is a minimizer of f. Assume that every subgradient used by the algorithm satisfies

$$\|g_k\| \le G, \qquad g_k \in \partial f(x_k),$$

for some constant $G > 0$. Consider the subgradient iteration

$$x_{k+1} = x_k - \alpha_k g_k,$$

with step sizes

$$\alpha_k = \frac{\alpha}{\sqrt{k+1}}, \qquad \alpha > 0.$$

(a) Show that for every $k \ge 0$,

$$\|x_{k+1} - x^\star\|^2 \le \|x_k - x^\star\|^2 - 2\alpha_k\big(f(x_k) - f(x^\star)\big) + \alpha_k^2 G^2.$$

(b) Deduce that for every $N \ge 0$,

$$\sum_{k=0}^{N} \alpha_k\big(f(x_k) - f(x^\star)\big) \le \frac{1}{2}\|x_0 - x^\star\|^2 + \frac{G^2}{2}\sum_{k=0}^{N} \alpha_k^2.$$

(c) Let

$$f_{\text{best}}^{(N)} := \min_{0 \le k \le N} f(x_k).$$

Show that

$$f_{\text{best}}^{(N)} - f(x^\star) \le \frac{\frac{1}{2}\|x_0 - x^\star\|^2 + \frac{G^2}{2}\sum_{k=0}^{N} \alpha_k^2}{\sum_{k=0}^{N} \alpha_k}.$$

(d) Using $\alpha_k = \alpha/\sqrt{k+1}$, show that

$$f_{\text{best}}^{(N)} - f(x^\star) = O\left(\frac{\log N}{\sqrt{N}}\right).$$

8. (10 points) Let

$$F(x) = f(x) + g(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable with L-Lipschitz gradient, and $g : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, closed, and convex. For $t > 0$, define the proximal operator

$$\mathrm{prox}_{tg}(v) := \arg\min_{x \in \mathbb{R}^n} \left\{ g(x) + \frac{1}{2t}\|x - v\|^2 \right\}.$$

Consider the proximal gradient iteration

$$x_{k+1} = \mathrm{prox}_{tg}\big(x_k - t\nabla f(x_k)\big), \qquad 0 < t \le \frac{1}{L}.$$

(a) Show that x_{k+1} minimizes the function

$$Q_k(x) := f(x_k) + \nabla f(x_k)^\top (x - x_k) + \frac{1}{2t}\|x - x_k\|^2 + g(x).$$

(b) Prove that for every k,

$$F(x_{k+1}) \le F(x_k) - \frac{1}{2}\left(\frac{1}{t} - L\right)\|x_{k+1} - x_k\|^2.$$

In particular, $F(x_k)$ is nonincreasing.

(c) Let $x^\star$ be a minimizer of F. Show that

$$F(x_{k+1}) - F(x^\star) \le \frac{1}{2t}\Big(\|x_k - x^\star\|^2 - \|x_{k+1} - x^\star\|^2\Big).$$

(d) Deduce that

$$F(x_N) - F(x^\star) \le \frac{\|x_0 - x^\star\|^2}{2tN} \qquad \text{for all } N \ge 1.$$

9. (10 points) Consider the primal linear program

$$\begin{aligned} \text{minimize } & c^\top x \\ \text{subject to } & Ax \geq b, \\ & x \geq 0, \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

(a) Derive the dual problem.

(b) Prove *weak duality*: if x is primal feasible and y is dual feasible, then

$$b^\top y \leq c^\top x.$$

(c) Suppose $x^\star$ and $y^\star$ are primal and dual feasible, respectively. Show that if

$$c^\top x^\star = b^\top y^\star,$$

then both are optimal.

(d) Show that if $x^\star$ and $y^\star$ are optimal, then the complementary slackness conditions hold:

$$y_i^\star (Ax^\star - b)_i = 0 \quad \text{for } i = 1, \ldots, m,$$

and

$$x_j^\star (c - A^\top y^\star)_j = 0 \quad \text{for } j = 1, \ldots, n.$$