Bubbling-Off Phenomena in Color TV Equation

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1 Introduction

The Color TV partial differential equation [1],

$$u_t = |\nabla u| \ \nabla \cdot \left(|\nabla u|^{-1} \nabla u \right),$$

is the unique variational geometric motion of second order suitable for regularizing features of codimension one, including all of their higher-codimensional junctions.

This fundamental equation can serve as a regularization replacing the ubiquitous heat equation, when preserving features such as edges and corners in an image is desired.

It was applied for this purpose to sustain Moore's Law in the microchip industry during 2000-2016, prior to the ability to perform EUV lithography at volume production, in order to regularize optical phase masks, which can be thought of as two dimensional vector fields with natural edges and corners.

Moreover, this equation can simulate mean-curvature "soap bubble" dynamics, when the discontinuous features are considered, with sufficiently high dimensions of the vector values.

As a geometric motion, this equation can be written as etiher an advection or a diffusion equation. We switch between these formulations freely below.

The solution theory of this PDE is still incomplete. This research note from 2005 shows that viscosity solution theory, which was applied to the 1-dimensional case of the original TV equation, is insufficient in the general case. In particular, we rigorously show, by exact calculation, the presence of bubbling-off phenomena. This should not be intuitively surprising, due to the ability of the equation to simulate soap bubble dynamics.

2 Flat Color TV

For maps $u : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ between flat spaces of equal dimension, the codimension-one color TV flow is based on the divergence of the generically $O(n)$-valued map $\operatorname{sgn}(\nabla u) := |\nabla u|^{-1} \nabla u$.

When $n = 1$, and thus $O(1) = \{\pm 1\}$, the resulting one-dimensional scalar TV flow is generically stationary, as we know. At local extrema of u, where the sign of ∇u changes, the flow is singular, and must be interpreted as a non-local evolution pulling in those local extrema.

For $n > 1$, it also true that at local extrema, where the orientation of u changes (that is, where $\operatorname{sgn}(\nabla u)$ changes between components of $O(n)$), the flow similarly has singular velocity pulling in the local extrema. Elsewhere, however, there is a meaningful finite flow that "untwists" the function.

For example, for $n = 2$ and u oriented, $\operatorname{sgn}(\nabla u)$ can be written as a rotation matrix with angle $\theta(x)$, and we find that values of u are advected with velocity $-\nabla \theta^\perp$, along the instantaneous level sets of θ. The result of this flow for $n = 2$ is therefore to make $\operatorname{sgn}(\nabla u)$ a constant rotation. Note that what is being made constant by this flow is not the angle of the levelest sets of u, but the angle between a levelest set of u and its image. In other words, for the steady-state solution, there is a global rotation of the target coordinates that makes every levelest set instantaneously parallel to its image.

For $n = 3$ and u oriented, writing

$$\operatorname{sgn}(\nabla u) = \exp \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$$

in an orthonormal target coordinate system chosen so that $\theta(0) = 0$, we find that the values of u are advected with velocity v with

$$v(0) = -(\nabla \times \theta)(0).$$

3 Bubbling-Off in Color TV

Due to its advectional constraint, Color TV can always be thought of as equidimensional, that is, an evolution of maps $u : X \longrightarrow Y$ with $\dim X =$

$\dim Y$. However, when going up in dimension, the image manifold Y usually has curved geometry. Even when u starts out as a smooth embedding into higher dimension, parts of Y with positive curvature can "bubble off," creating sgn-like discontinuities in finite time.

To provide a concrete example, let S be the sphere of radius $\frac{1}{\kappa}$ in $\mathbf{R}^3$, with "polar coordinates" at the north pole given by

$$(s,\phi) \mapsto \tfrac{1}{\kappa}\left(\sin(\kappa s)\cos\phi,\, \sin(\kappa s)\sin\phi,\, \cos(\kappa s)\right).$$

Put standard polar coordinates $(r,\theta) \mapsto r(\cos\theta,\, \sin\theta)$ on $\mathbf{R}^2$. Then define $u : \mathbf{R}^2 \longrightarrow S$ at time 0 by $\phi = \theta$ and $s = r$. We will show that the evolution preserves $\phi = \theta$, and evolves s as an odd analytic function of r up to time $t_0 = \frac{1}{\kappa}$, when the derivative of s at the origin blows up like $s'(0) = (1-\kappa t)^{-1/2}$.

The symmetries are clear. Let $\rho = \kappa s$ be the angle corresponding to s. Then we have

$$u = \tfrac{1}{\kappa}\left(\sin\rho\, \cos\theta,\, \sin\rho\, \sin\theta,\, \cos\rho\right),$$

with ρ an odd function of r depending also on t. Using the facts that

$$\begin{aligned}
r_x &= \cos\theta \\
r_y &= \sin\theta \\
(\cos\theta)_x &= \tfrac{1}{r}\sin^2\theta \\
(\cos\theta)_y = (\sin\theta)_x &= -\tfrac{1}{r}\cos\theta\, \sin\theta \\
(\sin\theta)_y &= \tfrac{1}{r}\cos^2\theta,
\end{aligned}$$

we can calculate the singular value decomposition of ∇u as

$$\begin{aligned}
\nabla u \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} &= \frac{\rho_r}{\kappa} \begin{bmatrix} \cos\rho\, \cos\theta \\ \cos\rho\, \sin\theta \\ -\sin\rho \end{bmatrix} \\
\nabla u \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} &= \frac{\sin\rho}{\kappa r} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
u_t &= |\nabla u|\, \nabla \cdot \left(|\nabla u|^{-1} \nabla u\right) \\
&= |\nabla u|\, \nabla \cdot \begin{bmatrix} 1 + (\cos\rho - 1)\cos^2\theta & (\cos\rho - 1)\cos\theta\, \sin\theta \\ (\cos\rho - 1)\cos\theta\, \sin\theta & 1 + (\cos\rho - 1)\sin^2\theta \\ -\sin\rho\, \cos\theta & -\sin\rho\, \sin\theta \end{bmatrix}
\end{aligned}$$

$$= \frac{\rho_r}{\kappa} \begin{bmatrix} \cos\rho\,\cos\theta \\ \cos\rho\,\sin\theta \\ -\sin\rho \end{bmatrix} \otimes \begin{bmatrix} \cos\rho\,\cos\theta \\ \cos\rho\,\sin\theta \\ -\sin\rho \end{bmatrix}^* \cdot \begin{bmatrix} \cos\theta\left(\frac{1}{r}(\cos\rho-1)-\rho_r\sin\rho\right) \\ \sin\theta\left(\frac{1}{r}(\cos\rho-1)-\rho_r\sin\rho\right) \\ -\frac{1}{r}\sin\rho-\rho_r\cos\rho \end{bmatrix}$$

$$= \frac{1-\cos\rho}{\kappa r}\rho_r \begin{bmatrix} \cos\rho\,\cos\theta \\ \cos\rho\,\sin\theta \\ -\sin\rho \end{bmatrix}.$$

On the other hand, calculating directly from the original formula for u,

$$u_t = \rho_t \begin{bmatrix} \cos\rho\,\cos\theta \\ \cos\rho\,\sin\theta \\ -\sin\rho \end{bmatrix}.$$

Thus,

$$\rho_t = \frac{1-\cos\rho}{\kappa r}\rho_r,$$

or back in terms of s,

$$s_t = \frac{1-\cos(\kappa s)}{\kappa r}s_r.$$

We will now attempt an odd series solution

$$s = \sum_{n \text{ odd}} s_n r^n,$$

with s_n functions of time only. The s_1 term obeys the equation

$$\dot{s}_1 = \tfrac{\kappa}{2}s_1^3$$

with solution

$$s_1 = \left(\kappa(t_0-t)\right)^{-1/2}.$$

The form of the recursive ODEs for s_n also suggests the substitution

$$s = \tfrac{1}{\kappa} f\left(\frac{\kappa^{1/2}r}{(t_0-t)^{1/2}}\right),$$

which reduces the above PDE for s to an algebraic equation for f, producing the following family of exact solutions:

$$s = \tfrac{2}{\kappa}\sin^{-1}\left(\tfrac{1}{2}\frac{\kappa^{1/2}r}{(t_0-t)^{1/2}}\right).$$

4 Singularities in Color TV (Positive Curvature Case)

Substituting $u = r^2$ in to the first-order PDE for s above, we can reduce it to the form

$$s_t + v(s)\, s_u = 0,$$

where

$$v(s) = -\tfrac{2}{\kappa}(1 - \cos(\kappa s)) = -\tfrac{4}{\kappa}\sin^2\left(\tfrac{\kappa}{2}s\right).$$

This type of advection equation can be solved exactly by the method of characteristics. The characteristics (along which s is constant) are straight lines (in (u,t)-space) with velocity $v(s(0,r))$. These characteristics hit $r = u = 0$ at time

$$\tau(r) = \frac{\kappa r^2}{4\sin^2\left(\frac{\kappa}{2}s(0,r)\right)}.$$

For the initial conditions $s(0,r) = r$ considered above, we find that $\tau(r)$ is a strictly increasing function on $r \in \left[0, \frac{2\pi}{\kappa}\right)$ with $\tau(0) = \frac{1}{\kappa}$ and $\tau'(0) = 0$. Therefore the singularity that begins at $t = \frac{1}{\kappa}$ and $r = 0$ is *not* a bubbling-off singularity. Instead, at $t = \frac{1}{\kappa}$ a kink forms at $r = 0$ with infinite derivative; the kink immediately turns into a sgn-like jump discontinuity which grows from zero size at $t = \frac{1}{\kappa}$ to removing the whole sphere at $t = \infty$.

Other types of singularities are possible. The specific family of solutions found previously, for example, shows bubbling-off, where the whole sphere disappears at once. But this requires a forcing boundary condition.

More generally, if $s(0,0) = 0$, $s_r(0,0) = 1$, and $s_{rrr}(0,0) = a$, then $\tau(r) = \frac{1}{\kappa}\left(1 - \frac{1}{3}(a - \frac{\kappa^2}{4})r^2\right) + O(r^3)$. If $a > \frac{\kappa^2}{4}$, but $s(0,r)$ eventually comes back close enough to r, then a circular shock will form at some $r > 0$, and will grow in height and migrate in to $r = 0$ at $t = \frac{1}{\kappa}$.

Applying viscosity solution theory to the above advection equation, we can consider initial conditions with shocks. Any "information-creating" shocks will be instantaneously smoothed into a cone of characteristics interpolating the shock values. "Information-destroying" shocks will evolve, and may be spontaneously created at any time. Indeed, because $v(s) < 0$ except on a discrete set of s values, any solution with $s(0,0) = 0$ and $s(0,r)$ non-constant must develop shocks in finite time.

References

[1] Paul Burchard. Total variation geometry I: Concepts and motivation. Report 02-01, University of California, Los Angeles, January 2002.