

Applied Differential Equations

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: _____

DATE: _____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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Pass/fail recommend on this form.

Total score: _____

Fall 2026, ADE Qual

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider axisymmetric heat diffusion in the circular disk, $r < 1$. The temperature field $u(r, t)$ satisfies the 2D heat equation in polar coordinates:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}, \quad r < 1, t > 0 \quad (1)$$

The boundary condition at the outer edge $r = 1$ is *convective*:

$$\frac{\partial u}{\partial r}(1, t) + u(1, t) = 0. \quad (2)$$

- (a) [3 points] Look for separable solutions $u(r, t) = R(r)T(t)$, showing that $T(t) = e^{-\lambda^2 t}$ (assume for now that $\lambda^2 > 0$), and deriving an appropriate equation for $R(r)$, for which you should state boundary conditions.
- (b) [3 points] By multiplying your differential equation from (a) by $rR(r)$, and integrating over $0 < r < 1$, show that $\lambda^2 > 0$.
- (c) [4 points]
 - (i) Making the change of variable $z = \lambda r$, deduce R satisfies *Bessel's equation of order zero*:

$$z^2 \frac{d^2 R}{dz^2} + z \frac{dR}{dz} + z^2 R = 0 \quad (3)$$

- (ii) Show that Eqn. 3 has exactly one independent solution that is regular in a neighborhood of $z = 0$. Call this solution $J_0(z)$.
 - (iii) Derive a transcendental equation for the eigenvalues, λ (your answer will involve the function J_0).

2. In this question, you will analyze two autonomous planar dynamical systems with solution families $(x(t), y(t)) \in \mathbb{R}^2$.

- (a) [5 points] Consider the system:

$$\begin{aligned} \dot{x} &= y - x^3, \\ \dot{y} &= -x^3 - y^3. \end{aligned}$$

By defining an appropriate Lyapunov function, prove that the origin $(0, 0)$ is globally asymptotically stable.

- (b) [5 points] Now consider a modified system where damping acts directly only on the y -variable:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x^3 - y^3. \end{aligned}$$

Prove that $(0, 0)$ is again globally asymptotically stable.

3. Consider the linear nonconstant coefficient PDE

$$u_t + xu_x = 0$$

with initial condition $u(x, 0) \equiv u_0(x) = 1$ for $x > x_0$ (a known constant) and zero elsewhere.

- (a) [3 points] Draw a characteristic diagram for this problem with x on the horizontal axis and t on the vertical axis. Derive the equation of a characteristic $x(t; X)$ starting at any point $(x, t) = (X, 0)$ on the x -axis.
- (b) [3 points] Write the solution of the PDE with the jump as the initial condition using the method of characteristics.
- (c) [4 points] Show that your solution, $u(x, t)$, solves the PDE for $t > 0$ in the sense of distributions.

4. Consider the *wave equation with spring force*

$$u_{tt} = c^2 u_{xx} + k u, \quad 0 < x < L, \quad t > 0, \quad (4)$$

in which $c > 0$, $L > 0$, and $k \in \mathbb{R}$ are constants. We impose boundary conditions $u(0, t) = u(L, t) = 0$ and initial data

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L. \quad (5)$$

- (a) [3 points] Use separation of variables to write the general solution of (4) as a series. Be explicit about how the time dependence of each mode depends on the sign of $k - c^2(n\pi/L)^2$.
- (b) [4 points] Determine the exact condition on (k, c, L) under which (4) admits solutions that grow exponentially in time. For such k , identify the fastest-growing mode, its growth rate, and the number of unstable modes.
- (c) [3 points] Define

$$E(t) = \frac{1}{2} \int_0^L \left(u_t^2 + c^2 u_x^2 - k u^2 \right) dx.$$

Show that $E(t)$ is independent of t for classical solutions of (4). Then, using the Poincaré inequality

$$\int_0^L u^2 dx \leq \left(\frac{L}{\pi} \right)^2 \int_0^L u_x^2 dx \quad \text{for } u \in H_0^1(0, L),$$

prove that if $k < c^2 \pi^2 / L^2$ then $\|u(\cdot, t)\|_{L^2(0, L)}$ is bounded uniformly in $t \geq 0$, with a bound depending only on $E(0)$, k , c , and L . Do not use the series from part (a).

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $\varepsilon > 0$, and let $W(u) = \frac{1}{4}(1 - u^2)^2$. Define the Ginzburg–Landau energy

$$E[u] = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx.$$

Throughout, all functions satisfy the homogeneous Neumann condition $\partial_\nu u = 0$ on $\partial\Omega$, and, where relevant, $\partial_\nu \Delta u = 0$ on $\partial\Omega$ as well.

- (a) [4 points] Compute the first variation of E , i.e. find $E'[u] \in L^2(\Omega)$ such that

$$\left. \frac{d}{ds} E[u + s\phi] \right|_{s=0} = \int_{\Omega} E'[u] \phi \, dx \quad \text{for all smooth } \phi \text{ with } \partial_{\nu} \phi = 0.$$

Show that the Allen–Cahn equation

$$u_t = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \tag{6}$$

is the $L^2(\Omega)$ gradient flow of E , and that the Cahn–Hilliard equation

$$u_t = \Delta \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} W'(u) \right) \tag{7}$$

is the gradient flow of E with respect to the H^{-1} inner product

$$\langle f, g \rangle_{H^{-1}} = \int_{\Omega} \nabla(-\Delta)^{-1} f \cdot \nabla(-\Delta)^{-1} g \, dx$$

on functions of zero mean, where $(-\Delta)^{-1}$ denotes the solution operator of the Neumann problem on mean-zero data. Derive the energy dissipation identity $\frac{d}{dt} E[u(t)] = -\mathcal{D}(t)$ for each equation, identifying $\mathcal{D}$ explicitly in each case.

- (b) [3 points] Show that solutions of (7) conserve the total mass $\int_{\Omega} u \, dx$, and that solutions of (6) in general do not. Exhibit initial data u_0 for which $\int_{\Omega} u \, dx$ is strictly monotone in t under (6), and explain the sign of $\frac{d}{dt} \int_{\Omega} u \, dx$.
- (c) [3 points] Prove that (6) preserves the bound $|u| \leq 1$: if u is a classical solution of (6) with $|u_0(x)| \leq 1$ for all $x \in \Omega$, then $|u(x, t)| \leq 1$ for all $x \in \Omega$ and $t \geq 0$.

6. Consider a static two-dimensional, incompressible fluid bounded by two parallel vertical walls at $x = -L$ and $x = L$. The upper free surface of the fluid is defined by $y = h(x)$. The total energy functional $J[h]$ contains the gravitational potential energy, liquid-gas surface energy, and wall-liquid surface energy:

$$J[h] = \int_{-L}^L \left(\frac{1}{2} \rho g [h(x)]^2 + \gamma \sqrt{1 + [h'(x)]^2} \right) dx - \gamma_{sl} (h(-L) + h(L)) \tag{8}$$

which contains 4 positive constants: ρ , fluid density, g , gravitational acceleration, γ , liquid-gas surface tension, and γ_{sl} , the wall-liquid surface tension. The total cross-sectional fluid volume is conserved:

$$A[h] = \int_{-L}^L h(x) \, dx \equiv A_0$$

- (a) [4 points] Incorporate the volume constraint using a Lagrange multiplier λ . Compute the first variation $\delta(J - \lambda A) = 0$ for arbitrary smooth variations $\eta(x)$, to derive a second order differential equation for h , which you do not need to solve.
- (b) [3 points] Allowing the end-point heights $h(\pm L)$ to vary freely, derive the boundary conditions for h at $x = \pm L$. Specifically, if the interface meets the solid wall at a contact angle θ_c , $h'(\pm L) = \pm \cot \theta_c$, derive the *Young equation*: $\cos \theta_c = \frac{\gamma_{sl}}{\gamma}$

- (c) [3 points] Assuming small surface deformations (specifically, $|h'(x)| \ll 1$), linearize the equation for $h(x)$. Show that $h(x)$ is approximately constant if $|x - L| \gg l_c$ and $|x + L| \gg l_c$ where $l_c \equiv \sqrt{\gamma/\rho g}$.

7. A thin liquid film of height $h(x, t) \geq 0$ rests on a flat substrate and is driven by a constant tangential stress applied at its free surface. In dimensionless variables the horizontal fluid velocity at height y above the substrate is

$$u(x, y, t) = y, \quad 0 \leq y \leq h(x, t),$$

so that u depends on x and t only through the range of y . The volume flux through a vertical cross-section of the film is

$$q(x, t) = \int_0^{h(x, t)} u \, dy,$$

and conservation of fluid volume gives

$$h_t + q_x = 0.$$

- (a) [3 points] Compute q as a function of h , and show that h satisfies the scalar conservation law

$$h_t + f(h)_x = 0, \quad f(h) = \frac{1}{2}h^2.$$

What is the characteristic speed $f'(h)$? How does it compare with the fluid velocity at the free surface $y = h$?

- (b) [4 points] Solve the Riemann problem

$$h(x, 0) = \begin{cases} h_l, & x < 0, \\ h_r, & x > 0, \end{cases}$$

for all $h_l, h_r \geq 0$. State the admissibility condition you use for discontinuous solutions and explain why it suffices for this flux. Show that a step *down* in thickness (thick film behind, thin film ahead) propagates as a shock, whereas a step *up* spreads out as a rarefaction wave. Explain this asymmetry in terms of the given velocity field $u = y$.

- (c) [3 points] Consider the shock connecting h_l to h_r with $h_l > h_r$, and let s denote its speed. Let

$$u_{\text{surf}} = u|_{y=h_l} \quad \text{and} \quad \bar{u} = \frac{1}{h_l} \int_0^{h_l} u \, dy$$

be, respectively, the velocity of fluid at the free surface of the thick film and the depth-averaged velocity of the thick film. Compute s , u_{surf} , and $\bar{u}$, and show that

$$\bar{u} \leq s < u_{\text{surf}},$$

with equality on the left exactly when $h_r = 0$. Explain, using conservation of volume, why the front advances at the speed s and not at the surface speed.

8. The λ - ω reaction-diffusion system is used as a model for oscillatory traveling waves in biology and chemistry. In one spatial dimension, it models two reacting chemicals via real-valued fields $u(x, t)$ and $v(x, t)$:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \lambda(r)u - \omega(r)v, \quad (9)$$

$$\frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + \omega(r)u + \lambda(r)v, \quad (10)$$

where $r = \sqrt{u^2 + v^2} > 0$, $D > 0$ is a scalar diffusion coefficient, and $\lambda(r), \omega(r) \in C^1(\mathbb{R}^+)$ are smooth functions. Assume $\lambda(1) = 0$ with $\lambda'(1) < 0$, and $\omega(1) \equiv \omega_1 > 0$.

- (a) [4 points] Assuming spatial homogeneity ($\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = 0$), transform the ODE system into polar coordinates (r, θ) . Prove that $r = 1$ corresponds to a unique, asymptotically stable limit cycle in the phase space.
- (b) [6 points] Consider the spatially extended PDE. Look for traveling plane wave solutions of the form:

$$u(x, t) = R \cos(\Omega t - kx), \quad v(x, t) = R \sin(\Omega t - kx),$$

where $R > 0$ is a constant wave amplitude, k is the spatial wavenumber, and Ω is the wave frequency.

- (1) Derive the algebraic conditions governing R and Ω as functions of k and D .
- (2) For the specific case $\lambda(r) = 1 - r^2$ and $\omega(r) = \omega_0 - \beta r^2$ (with $\beta \in \mathbb{R}$), calculate $R(k, D)$ and $\Omega(k, D)$ explicitly. Determine the critical wavenumber range k for which physical plane wave solutions exist, and describe how wave existence and amplitude depend on the diffusivity, D .