

Analysis

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: _____

DATE: _____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

1. _____

5. _____

2. _____

6. _____

3. _____

7. _____

4. _____

8. _____

Pass/fail recommend on this form.

Total score: _____

ANALYSIS QUALIFYING EXAM: SEPTEMBER 19, 2026

In order to pass the exam, you need to show mastery of both real and complex analysis. Choose ten problems to work on, including five from Problems 1–6 and five from Problems 7–12. Please indicate on the front of your paper which ten problems you wish to have graded.

Problem 1. Let $p > 1$ and let $\{f_n\}$ be a sequence in $L^p(\mathbb{R}) \cap C^1(\mathbb{R})$. Suppose that $f_n(0) = 0$ for all $n \in \mathbb{N}$ and that the sequence $\{f'_n\}$ is bounded in $L^p(\mathbb{R})$.

(a) Show that a subsequence of $\{f_n\}$ converges locally uniformly to a uniformly Hölder continuous function f . Find the Hölder exponent of f .

(b) Show that a subsequence of $\{f'_n\}$ converges weakly to a function g in $L^p(\mathbb{R})$.

(c) For $p = 3$, give an example of a sequence $\{f_n\}$ for which $\{f'_n\}$ does not have convergent subsequence in $L^3(\mathbb{R})$.

Problem 2. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative, nonzero function. Show that the set $\mathcal{S}$ of linear combinations of functions $\{\phi(\frac{x-a}{b}), a \in \mathbb{R}^d, b > 0\}$ is dense in $L^1(\mathbb{R}^d)$.

Problem 3. Let (Ω, Σ, μ) be a measure space and ν be a Borel measure on $[0, \infty)$ such that $\phi(t) := \nu([0, t]) < \infty$ for every $t > 0$. Show that for each nonnegative measurable function f on Ω the following statements are true:

(a) the function $F(x, t) := \chi_{\{f > t\}}(x)$ is measurable on $\Omega \times [0, \infty)$ with respect to the product measure $\mu \times \nu$;

(b) we have $\int_{\Omega} \phi(f(x)) \mu(dx) = \int_0^{\infty} \mu(\{x : f(x) > t\}) \nu(dt)$.

Problem 4. (a) Let $\{a_n\}$ be a sequence in $[0, 1]$. We define a linear functional on ϕ on $C([0, 1])$ by setting

$$\phi(f) := \sum_{n=1}^{\infty} 2^{-n} f(a_n)$$

for $f \in C([0, 1])$.

Show that there is no function $g \in L^1([0, 1])$ such that

$$\phi(f) = T_g(f) := \int_0^1 f(x)g(x)dx$$

for all $f \in C([0, 1])$.

(b) Show that there are continuous functionals on $L^\infty([0, 1])$ that are **not** of the form T_g for some $g \in L^1([0, 1])$.

Problem 5. Let Ω be a bounded domain in $\mathbb{R}^d$, and let $\mathcal{S}$ be a closed and convex subset of $L^4(\Omega)$. Show that there exists a unique minimizer in $\mathcal{S}$ of the energy functional

$$E(f) := \int_{\Omega} |f|^4 dx.$$

Problem 6. We use the notation $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$ for $r > 0$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and suppose that there is $\lambda > 0$ such that

$$Mf(x) := \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy \geq \lambda \text{ for } x \in B_1.$$

Show that that then

$$Mf(x) \geq c\lambda \text{ for } x \in B_2,$$

where $c = c(d) > 0$ is a constant only depending on d .

Problem 7. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. We consider entire solutions w of the differential equation

$$w'(z) = f(w(z)), \quad z \in \mathbb{C}. \quad (1)$$

(a) Suppose $w_1: \mathbb{C} \rightarrow \mathbb{C}$ and $w_2: \mathbb{C} \rightarrow \mathbb{C}$ are two entire solutions of (1). Show that if there exists a point $z_0 \in \mathbb{C}$ with $w_1(z_0) = w_2(z_0)$, then $w_1 = w_2$.

(b) Suppose (1) has two distinct entire solutions w_1 and w_2 . Show that then every entire solution w of (1) can be represented in the form $w = w_1 + c(w_2 - w_1)$, where $c \in \mathbb{C}$ is a constant.

Hint: Use w, w_1, w_2 to construct an auxiliary function and show that it is constant.

Problem 8. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function satisfying

$$|f(z)| \leq e^{\frac{1}{2}|z|^2}$$

for all $z \in \mathbb{C}$. Show that then

$$\limsup_{n \rightarrow \infty} \sqrt{n} |a_n|^{1/n} < \infty.$$

Problem 9. Let $f: \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \rightarrow \mathbb{D}$ be a nonconstant holomorphic function satisfying $|f(z_n)| \rightarrow 1$ for every sequence $\{z_n\}$ in $\mathbb{D}$ with $|z_n| \rightarrow 1$ as $n \rightarrow \infty$.

(a) Show that if $w \in \mathbb{D} \cap \overline{f(\mathbb{D})}$, then $w \in f(\mathbb{D})$.

(b) Show that $f(\mathbb{D}) = \mathbb{D}$.

Problem 10. Consider the function $f(z) = z - 2 - e^{-z}$.

(a) Show that f has a unique zero z_0 on the positive real axis.

(b) Show that z_0 is the only zero of f in the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

Problem 11. Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function with $f(z) \neq 0$ for $z \in \mathbb{D}$. Show that then $|f'(0)| \leq 2/e$. Hint: Reduce to the case $0 < f(0) < 1$ and argue that you can write f in the form $f = e^{-g}$ with a holomorphic function g .

Problem 12. Let $u: \mathbb{C} \rightarrow \mathbb{R}$ be a harmonic function satisfying $u(z+1) = u(z)$ for $z \in \mathbb{C}$. Show that then there exist a constant $c \in \mathbb{R}$ and a holomorphic function $f: \mathbb{C} \rightarrow \mathbb{C}$ such that $u(z) = \operatorname{Re}(f(z))$ and $f(z+1) = f(z) + ic$ for $z \in \mathbb{C}$.