

BASIC EXAM: FALL 2026

Test instructions:

- Write your UCLA ID number on the upper right corner of each page.
- **Do not write your name anywhere on the exam!!!**
- All answers must be justified. If you wish to use a known theorem, make sure to give a full and precise statement.
- Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems.
- Clearly indicate below which 10 problems you wish to have graded.
- Please staple your problems in numerical order.
- **Important:** No books, notes, calculators, smart phones, computers or other printed or electronic materials can be used on the exam.

1	2	3	4	5	6
7	8	9	10	11	12

Analysis problems

Problem 1. Let A be an $n \times n$ matrix with complex entries and let

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

be the exponential series in powers of A .

- (a) Show that for all such matrices A , this series converges absolutely in the Frobenius norm (also known as the Euclidean norm), and that this convergence is uniform on any compact set of matrices with complex entries.
- (b) Assume that B is another $n \times n$ matrix with complex entries, for which $AB = BA$. Prove that

$$\exp(A + B) = \exp(A) \exp(B).$$

Problem 2. Let $X = \{f: [0, 1] \rightarrow [0, 1] \mid f \text{ continuous, } f(1) = 0\}$. Define a distance function d on X by

$$d(f, g) = \inf\{a \in [0, 1] \mid f(x) = g(x) \text{ for all } x \in [a, 1]\}.$$

Prove that d is a metric on X and that the metric space (X, d) is neither separable nor complete.

Problem 3. Let f be a real-valued function with domain $\mathbb{R}$, and suppose that for all $x \in \mathbb{R}$,

$$\lim_{z \rightarrow x} f(z) \text{ exists.}$$

Prove that the function h defined by $h(x) = \lim_{z \rightarrow x} f(z)$ is continuous on $\mathbb{R}$.

Problem 4. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with $\text{Dom}(f) = \mathbb{R}$. For each positive integer n , define $h_n: \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_n(x) = n[f(x + \frac{1}{n}) - f(x)].$$

Prove that $h_n \rightarrow f'$ uniformly on $[0, 1]$.

Problem 5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that for all real numbers $a < b$,

$$\forall n \geq 1 : \int_a^b f(x) x^n dx = 0.$$

Prove that $f = 0$.

Problem 6. Consider the system of equations

$$\begin{aligned}x^2y + y^2z + z^2x &= 2, \\xe^y + z^3 &= 3, \\x^2 - 2y + z^2 &= f(x),\end{aligned}$$

in a neighborhood of $(2, 0, 1)$. Prove that there exists a unique differentiable real-valued function f , defined on an open interval containing $x = 2$, that satisfies the third equation. Then compute $f'(2)$.

Linear algebra problems

Problem 7. Let V be an inner product space, and let $T: V \rightarrow V$ be a linear operator. Show that the orthogonal complement of the image of T is the kernel of the adjoint T^* ; in other words, show that $\text{im}(T)^\perp = \ker(T^*)$.

Problem 8. For an $n \times n$ matrix M , its commutant $C(M)$ is the vector space of all matrices N such that $MN = NM$. Suppose that the Jordan decomposition of M consists of a single Jordan block. Determine the dimension of $C(M)$ and find a basis for it.

Problem 9. For a square matrix M with complex entries, show that $\exp(M)$ (defined as in Problem 1) is diagonalizable if and only if M is diagonalizable.

Problem 10. Let V be a finite-dimensional vector space over a field F , let T be a diagonalizable linear operator on V , and let $p(x)$ be a minimal polynomial of T . Prove that there exists some $v \in V$ such that the subspace

$$\text{span}(\{v, T(v), T^2(v), T^3(v), \dots\})$$

has dimension equal to $\deg(p)$.

Problem 11. Let X be a finite set of cardinality n , and let V be the vector space of all real-valued functions $f: X \rightarrow \mathbb{R}$. Let W be any subspace of V , and let $k = \dim(W)$. Prove that there exist distinct $x_1, \dots, x_k \in X$ and functions $f_1, \dots, f_k \in W$ such that

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for all } i, j \in \{1, \dots, k\}.$$

Problem 12. The $n \times n$ matrix A with real entries is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}| \text{ holds for each } i = 1, 2, \dots, n.$$

Show that if A is strictly diagonally dominant, then the linear system $Ax = b$ has a unique solution $x \in \mathbb{R}^n$ for any vector $b \in \mathbb{R}^n$. Justify your answers.