

Geometry/Topology

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: _____

DATE: _____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

1. _____

5. _____

2. _____

6. _____

3. _____

7. _____

4. _____

8. _____

Pass/fail recommend on this form.

Total score: _____

QUALIFYING EXAM
Geometry/Topology
September 2026

Attempt all ten problems. Each problem is worth 10 points. You must fully justify your answers. In order to pass, you must receive 60/100 AND (get 8/10 on 6 problems OR (get 8/10 on 5 problems AND get 5/10 on 2 other problems)).

You may use the conclusion of one problem to solve later problems, even if you do not yet have a complete proof of that conclusion. In particular, you may use the conclusion of a subpart of a problem to solve a later subpart, without having a complete proof of that conclusion.

1. State the classical Stokes' theorem for a compact, oriented 2-dimensional submanifold of $\mathbb{R}^3$ with smooth boundary. Derive it from Stokes' theorem for differential forms.
2. The Grassmannian $\text{Gr}(k, n)$ is the set of all k -dimensional subspaces of $\mathbb{R}^n$. Explicitly construct the structure of a smooth manifold on $\text{Gr}(k, n)$ using atlases and transition functions. What is its dimension?
3. Let $x_1, x_2, \dots, x_k$ and $y_1, y_2, \dots, y_k$ be two sets of distinct points in a connected smooth manifold M of dimension at least 2, and $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_k$ be the corresponding two sets of non-zero tangent vectors at these points. Show that there is a diffeomorphism f of M such that $f(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$ for $i = 1, \dots, k$.
4. Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactly supported n -form. Show that $\omega = d\eta$ for some compactly supported $(n-1)$ -form $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.
5. Let

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\}) / (x_0, \dots, x_n) \sim t(x_0, \dots, x_n),$$

for all $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} - \{0\}$ and $t \in \mathbb{R} - \{0\}$ be the real n -dimensional projective space, and let $X = \{[(x_0, \dots, x_n)] \in \mathbb{RP}^n \mid x_0 = 0\}$, where $[(x_0, \dots, x_n)]$ is the equivalence class of $(x_0, \dots, x_n)$. Is it possible to find a smooth map $f: \mathbb{RP}^n \rightarrow \mathbb{R}$ with $0 \in \mathbb{R}$ as a regular value and preimage $f^{-1}(0) = X$?

6. A map $p: E \rightarrow B$ is a *fiber bundle with fiber F* if every $b \in B$ has an open neighborhood U and a homeomorphism $p^{-1}(U) \cong U \times F$ under which p is projection onto U . Assume that B and F are path-connected, choose $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$, and identify $p^{-1}(b_0)$ with F .

(a) (4 pts) Prove that

$$\pi_1(F, e_0) \longrightarrow \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0) \longrightarrow 1$$

is exact.

- (b) (3 pts) Further assume that B and F are finite CW complexes and that E is Hausdorff. Prove that E has the homotopy type of a finite CW complex $\widehat{E}$ with one cell $e_\alpha^p \times f_\beta^q$ of dimension $p + q$ for each p -cell $e_\alpha^p \subset B$ and q -cell $f_\beta^q \subset F$. Deduce that, for each n ,

$$C_n^{\text{cell}}(\widehat{E}) \cong \bigoplus_{p+q=n} C_p^{\text{cell}}(B) \otimes C_q^{\text{cell}}(F),$$

as an abelian group.

(Hint: You may use the fact that any fiber bundle can be trivialized over a disk in the base space and the fact that any map between CW complexes can be homotoped to a cellular map.)

- (c) (3 pts) Prove that

$$\chi(E) = \chi(B)\chi(F).$$

7. Let $k, m, n \geq 1$, and suppose that $S^n \xrightarrow{p} S^m$ is a fiber bundle with fiber S^k .

- (a) (2 pts) Prove that $n = m + k$.

- (b) (4 pts) Prove that $|m - k| = 1$.

(Hint: You may use the conclusion of Problem 6(b).)

- (c) (4 pts) Prove that $m - k = 1$.

8. Reuse the set-up from Problem 7.

- (a) (5 pts) Let $C_p = S^m \cup_p CS^n$ be the mapping cone of p . Prove, as a graded ring,

$$H^*(C_p; \mathbb{Z}) \cong \mathbb{Z}[U]/(U^3),$$

where U is in grading m .

- (b) (5 pts) Prove that k is odd. (Remark: Such bundles exist for $k = 1, 3$, but not for $k = 5$.)

9. Let $V = \mathbb{R}^7$ have standard basis $e_1, \dots, e_7$ and dual basis $e^1, \dots, e^7$. Write

$$e^{i_1 \dots i_r} = e^{i_1} \wedge \dots \wedge e^{i_r}, \quad \text{vol} = e^{1234567},$$

and put

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

and

$$G = \{A \in GL(V) \mid A^* \varphi = \varphi\}.$$

Consider $\Phi: GL(V) \rightarrow \Lambda^3 V^*$, $\Phi(A) = A^* \varphi - \varphi$.

- (a) (2 pts) Prove that $\dim \ker(d\Phi_A) = \dim \ker(d\Phi_I)$ for all $A \in G$.

- (b) (3 pts) Prove

$$\ker(d\Phi_I) \subset \mathfrak{so}(7) = \{X \in M_7(\mathbb{R}) \mid X^T + X = 0\}.$$

- (c) (3 pts) Prove $\dim \ker(d\Phi_I) = 14$.
 - (d) (2 pts) Prove that G is a smooth submanifold of $GL(V)$ and compute its dimension.
10. Let G be the smooth manifold constructed in Problem 9. Suppose that we also know G is a compact Lie group.
- (a) (4 pts) By considering the action of the special unitary group $SU(n)$ on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$, prove that there is a fiber bundle

$$SU(n-1) \longrightarrow SU(n) \xrightarrow{p} S^{2n-1}.$$

- (b) (3 pts) Suppose that by considering the action of G on the unit sphere $S^6 \subset \mathbb{R}^7$, we know there is fiber bundle

$$SU(3) \longrightarrow G \xrightarrow{p} S^6.$$

Compute $\pi_1(G)$.

(Hint: You may use the conclusion of Problem 6(a).)

- (c) (3 pts) Compute $H_*(G; \mathbb{Z})$ for $* \leq 3$.

(Hint: You may use the conclusion of Problem 6(b).)