

Logic

INSTRUCTIONS FOR QUALIFYING EXAMS

Start each problem on a new sheet of paper. Write your university identification number at the top of each sheet of paper. **DO NOT WRITE YOUR NAME!**

Complete this sheet and staple to your answers. Read the directions of the exam carefully.

STUDENT ID NUMBER: _____

DATE: _____

EXAMINEES: DO NOT WRITE BELOW THIS LINE

1. _____

5. _____

2. _____

6. _____

3. _____

7. _____

4. _____

8. _____

Pass/fail recommend on this form.

Total score: _____

Please answer all questions. You must prove all your answers, even when this is not explicitly requested. In each problem, the level of details you give and your choice of which standard results to prove and which to use without proof should be appropriate to the question; you have to demonstrate that you know the arguments relevant to the question. For each problem item, you may use previous problems and items if needed, even if you did not answer them.

For a formula φ , $|\varphi|$ denotes its length.

Problem 1. Let K be an algebraically closed field, viewed as a structure in the usual language $L_{\text{Ring}} = \{0, 1, +, -, \cdot\}$ of rings. Let $\bar{a} \in K^n$ be a tuple of parameters and let $\varphi(x, y, \bar{a})$ be an L_{Ring} -formula with free variables x, y that defines a function $f: K \rightarrow K$. Suppose that the function f is *additive*, i.e., for all $x, y \in K$, we have

$$f(x + y) = f(x) + f(y).$$

(1a) Show that if $\text{char}(K) = 0$, then there is $c \in K$ such that $f(x) = cx$ for all $x \in K$.

(1b) Obtain the same conclusion when $\text{char}(K)$ is sufficiently large as a function of $|\varphi|$.

Problem 2. Let $\mathcal{M}$ be a structure in a countable language L and suppose that $\text{Th}(\mathcal{M})$ is $\aleph_0$ -categorical. Prove that every injective definable function $f: M \rightarrow M$ is surjective.

Problem 3. Prove that a non-empty set $X \subseteq \mathbb{N}$ is recursive if and only if it is the image of a nondecreasing total recursive function.

Problem 4. Let $\varphi_e, e \in \mathbb{N}$, be the standard recursive enumeration of the recursive functions. Call a (possibly partial) function f on $\mathbb{N}$ *self-coding* if $f(0) \downarrow$ and f is equal to $\varphi_{f(0)}$.

(4a) Prove that there exists a total self-coding function.

(4b) Classify the set $U := \{e \in \mathbb{N} : \varphi_e \text{ is self-coding}\}$ in the arithmetical hierarchy.

Problem 5. Show that there exists a model $\mathcal{N}$ of PA with a nonstandard element $a \in N$ that is definable in $\mathcal{N}$ without parameters.

Problem 6. For $f, g: \mathbb{N} \rightarrow \mathbb{N}$, write $f <^* g$ if $f(n) < g(n)$ for all sufficiently large $n \in \mathbb{N}$. Show that there is a sequence $(f_\alpha)_{\alpha < \omega_1}$ of functions such that $f_\beta <^* f_\alpha$ for all $\beta < \alpha < \omega_1$.

Problem 7. Assume $V = L$. Let θ be a regular uncountable cardinal and let $p \subseteq L_\theta$ be finite. Let H consist of all $x \in L_\theta$ that are definable in L_θ with parameter p . Show that $H \preceq L_\theta$.

Problem 8. Assume $V = L$. Let κ be a regular uncountable cardinal and let S be the set of all $\alpha < \kappa$ for which there exist an ordinal $\beta > \alpha$ and a finite set $p \subseteq L_\beta$ such that:

- (1) $L_\beta \models \text{ZFC} - \text{Powerset} + \text{'}\alpha \text{ is regular uncountable'}$, and
- (2) if $X \preceq L_\beta$, $p \subseteq X$, and $X \cap \alpha$ is an ordinal, then $X = L_\beta$.

Prove that S is stationary.