

Name: _____
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UCLA MATHEMATICS - NUMERICAL ANALYSIS: FALL 2026

INSTRUCTIONS and GRADING:

There are 7 problems. Problems 1-2 are worth 5 points and problems 3-7 are worth 10 points. All problems will be graded and counted towards the final score. You have to demonstrate a sufficient amount of work on both groups of problems $[1 - 4]$ and $[5 - 7]$ to obtain a passing score.

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM. You have 4 hours.
Good luck.

NOTATION:

#	Score	Points Possible
1	_____	5
2	_____	5
3	_____	10
4	_____	10
5	_____	10
6	_____	10
7	_____	10
Total	_____	60

1. [5 Pts.] For integers $n \geq 0$, let

$$I_n = \int_0^1 \frac{x^n}{x+2} dx.$$

- (a) Show that $I_n = 1/n - 2I_{n-1}$ for $n \geq 1$.
 (b) Starting with $\widehat{I}_0 = I_0 + \delta_0$, where δ_0 is the initial error, define

$$\widehat{I}_n = \frac{1}{n} - 2\widehat{I}_{n-1}, \quad n \geq 1.$$

Find $\delta_n = \widehat{I}_n - I_n$ explicitly. Does this recurrence amplify or reduce the starting error?

- (c) Fix integers $N > m \geq 0$. Starting instead with $J_N = 0$, compute backward using

$$J_{n-1} = \frac{1}{2n} - \frac{1}{2}J_n, \quad n = N, N-1, \dots, m+1.$$

Prove that

$$|J_m - I_m| \leq \frac{2^{-(N-m)}}{2(N+1)}.$$

What happens to this bound as $N \rightarrow \infty$ with m fixed?

2. [5 Pts.] Suppose $f : [0, 1] \rightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y| \quad \text{for all } x, y \in [0, 1].$$

We approximate $\int_0^1 f(x) dx$ by the one-point quadrature rule $Q_c(f) = f(c)$, where $c \in [0, 1]$ is chosen independently of f .

- (a) Prove that

$$\left| \int_0^1 f(x) dx - f(c) \right| \leq \frac{c^2 + (1-c)^2}{2}.$$

- (b) For each c , give a function for which equality holds in part (a).
 (c) Determine the value of c that minimizes the largest possible quadrature error over all functions satisfying the stated condition. Give this minimum error.

3. [10 Pts.] Consider the equation

$$x = e^{-x}.$$

We will compare the direct fixed-point iteration with Newton's method for solving this equation.

- (a) Show that the equation has a unique solution $\alpha \in (0, 1)$.
 (b) Let $g(x) = e^{-x}$ and $H(x) = g(g(x))$. Show that g maps $[0, 1]$ into itself and that H is contractive. For the iteration $x_{n+1} = g(x_n)$ with any $x_0 \in [0, 1]$, deduce that

$$|x_{2m} - \alpha| \leq e^{-m}|x_0 - \alpha|, \quad m = 0, 1, 2, \dots$$

Explain why the entire sequence x_n converges to α .

- (c) Apply Newton's method to $f(x) = x - e^{-x}$, starting from $y_0 = 0$. Write it as a fixed-point iteration $y_{n+1} = G(y_n)$ and simplify G to an explicit formula. Prove that, for every $n \geq 0$,

$$0 \leq y_n \leq y_{n+1} \leq \alpha, \quad \alpha - y_{n+1} \leq \frac{1}{4}(\alpha - y_n)^2.$$

Deduce that $y_n \rightarrow \alpha$.

4. [10 Pts.] Consider

$$y'(t) = \lambda y(t), \quad y(0) = 1, \quad \lambda \in \mathbb{C}.$$

Let $T > 0$ be fixed, $h = T/N$, and $t_n = nh$ and approximate the ODE using the Backward Euler method.

- (a) Find the amplification factor $R(z)$, defined by $u_{n+1} = R(h\lambda)u_n$. Prove that the method is A-stable, meaning that $|R(z)| \leq 1$ whenever $\operatorname{Re} z \leq 0$.
 (b) For this and the remaining part, take $\lambda = -1$. Find u_n explicitly and prove that

$$\max_{0 \leq n \leq N} |u_n - e^{-t_n}| \leq C_T h$$

for all sufficiently small h , where C_T is independent of N .

- (c) Suppose the computed values satisfy

$$v_0 = 1, \quad (1 + h)v_{n+1} = v_n + r_{n+1}, \quad n = 0, \dots, N-1.$$

Here r_{n+1} measures the error in satisfying the equation at step $n+1$. Assume $|r_j| \leq \tau_h$ for $j = 1, \dots, N$, where $\tau_h \geq 0$. Prove that

$$\max_{0 \leq n \leq N} |v_n - u_n| \leq \frac{\tau_h}{h} [1 - (1 + h)^{-N}].$$

Show that, if $\tau_h = h^2$,

$$\max_{0 \leq n \leq N} |v_n - e^{-t_n}| = O(h)$$

as $N \rightarrow \infty$, for every error sequence satisfying the stated bound.

5. [10 Pts.] Consider the initial value problem

$$u_t + au_x = \varepsilon u_{xx}, \quad 0 \leq x \leq 1, \quad t > 0,$$

where $a > 0$ and $\varepsilon > 0$ are constants. Assume that $u(x, 0) = u_0(x)$, u_0 is smooth, and impose periodic boundary conditions $u(x+1, t) \equiv u(x, t)$.

- (a) Using Crank-Nicolson, construct a finite difference scheme for the above problem that is second-order accurate in both space and time. Determine a stability condition for your scheme.
 (b) Determine what happens to your scheme as $\varepsilon \rightarrow 0^+$. Does the limiting scheme remain stable and convergent for the pure advection equation $u_t + au_x = 0$?

Justify all of your answers.

6. [10 Pts.] Consider the initial value problem

$$u_t + \frac{\partial}{\partial x}(u(1-u)) = 0,$$

to be solved for $0 \leq x \leq 1$, with smooth periodic initial data $u(x, 0) = u_0(x)$, and periodic boundary conditions $u(x+1, t) = u(x, t)$. Assume that $0 \leq u_0(x) \leq 1$.

Construct a finite difference scheme that is convergent as $\Delta x, \Delta t \rightarrow 0$. Justify your scheme by deriving a condition on Δx and Δt under which the scheme is monotone and L^∞ -stable.

7. [10 Pts.] Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with

$$\partial\Omega = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset,$$

where Γ_0 has positive $(n-1)$ -dimensional measure. Consider the boundary value problem

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + b(x)u = f(x) \quad \text{in } \Omega,$$

subject to

$$u = 0 \quad \text{on } \Gamma_0,$$

and

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} n_i = 0 \quad \text{on } \Gamma_1.$$

Assume that $a_{ij}, b \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and that there exist constants $\alpha > 0$ and $b_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega,$$

and

$$b(x) \geq b_0 \quad \text{for a.e. } x \in \Omega.$$

- Give a weak formulation of the boundary value problem and prove that it has a unique weak solution using the Lax-Milgram theorem.
- Describe a Galerkin finite element method using piecewise linear elements on a triangulation of Ω . Explain how the boundary conditions are imposed and how you would solve the linear system. Justify why the linear system has a unique solution.
- Assuming that $u \in H^2(\Omega)$, provide an estimate for

$$\|u - u_h\|_{H^1(\Omega)}$$

in terms of the mesh size h . State any assumptions on the triangulations you used. Justify all of your answers.